# Certainty and Uncertainty in the Taxation of Risky Returns

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## Abstract

I extend the general equilibrium techniques that have been applied to proportionate taxes to analyze the economic impact of non-proportionate taxes, including those with such commonly observed features as loss disallowances and progressivity. I analogize proportionate taxes to financial forwards and more general taxes to structured financial options. Option pricing theory and methods carry over naturally, and in general the burden of an income tax has a "certainty equivalent value" equal to the price of a corresponding complex option. A direct consequence is that non-proportionate taxes specifically burden the risk in risky investment returns, but not the expected level of these returns. If expected return levels are determined as a function of risk, then they may be burdened indirectly, but the existence of such a functional relationship depends on particular choices of market model and portfolio, such as an investment in the market portfolio under the CAPM.

After developing the general theory, I apply it to the specific example of a tax that is proportionate on gains but disallows loss offsets. Such a tax burdens the risk of returns and therefore encourages portfolio diversification targeted at risk minimization, without regard to expected portfolio return. It also heavily burdens investment in put and call options, with such investments being more disfavored the more out of the money, and hence the more risky, the options are. In addition, it penalizes division of risky asset ownership across taxpayers using put-call parity, and interestingly this means that it penalizes the use of debt financing as well. Finally, I describe ways in which the general theory applies both to broader classes of taxes and also to particular aspects of the current U.S. tax system.

**Keywords**: Income Taxation; Proportionate Income Taxation; Loss Offsets; Progressivity; Risky Return to Assets; Domar-Musgrave; General Equilibrium

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# 1 Introduction

It is well understood in the law and public finance literature that a constant-rate tax levied proportionately on investment income is equivalent to a tax on the risk-free return to initial investment wealth, at least under idealized assumptions about the supply of assets, the interest rate for borrowed money, and the de minimis nature of transaction costs.[1] What is less well understood is how precisely this result changes when the tax is not proportionate, including when the tax has such commonly observed features as loss disallowances and progressivity. This paper addresses the gap in understanding by extending the general equilibrium approach developed by Kaplow (1991, 1994) to encompass continuous-time trading and then using option pricing techniques to analyze systematically non-proportionate methods of taxation.

The central theoretical finding is that the tax levied on an asset's return under a non-proportionate system of taxation can be translated into an equivalent up-front lump-sum tax and that this "certainty equivalent value" can be calculated in a precise and systematic way as the price of a specific financial option. A direct consequence is that non-proportionate taxes burden the risk in risky returns, but not the expected level of these returns, just as option prices depend on underlying risk but not on expected return levels. If expected return levels are determined as a function of risk, then they may be burdened indirectly. However, the existence of such a functional relationship between risk and return depends on assumptions about the model describing returns and the particular choice of portfolio. For example, return is proportionate to risk for the market portfolio in the Capital Asset Pricing Model (CAPM), but this is not generally the case for other portfolios or in multi-factor models of asset returns.

After the theoretical model is developed, it is applied to study in detail the case of a tax that is proportionate for gains but does not allow offsets for losses. This system is a convenient choice for initial study because it is only slightly more complex than a proportionate tax, and because many real-world taxes do in fact deny offsets for some or all losses.[2] As expected from the general theory, this tax is found to burden the expected return on a risky investment and to encourage diversification targeted at risk minimization, without regard to expected portfolio return. In addition, it is found to burden significantly

[1] Domar and Musgrave (1944, 1945) lay out the result in its original form, and Kaplow (1991, 1994) extends it to a general equilibrium setting. Additional discussion and development in the legal literature can be found, for example, in Warren (1996) and Weisbach (2004, 2005). See Avi-Yonah (2004) for a discussion of some of the limitations resulting from borrowing costs and transaction costs.

[2] Under §1211 of the Internal Revenue Code, for example, net capital losses are disallowed to corporations and allowed only to the extent of $3,000 per year for individuals.

the ownership of put and call options, with the burden growing larger the farther out of the money, and hence more levered and risky, the options are. It also penalizes synthetic division of risky asset ownership across taxpayers using put-call parity. Interestingly, this means that it penalizes the use of debt financing: that is, a lower aggregate burden is imposed on two investors who hold direct interests in a risky asset than is imposed on one investor who holds a debt claim on the asset and another who holds an equity claim.

Finally, after investigating the proportionate tax that does not allow loss offsets, properties of certain other classes of tax are investigated, including convex and concave taxes[3] and subadditive and superadditive taxes.[4] Taxes that are convex tend to burden risk-taking more heavily, and those that are concave tend to subsidize risk-taking (relative to risk-free investments). Taxes that are subadditive tend to discourage division of ownership, and those that are superadditive tend to encourage division of ownership. The current U.S. tax system exhibits some properties of each of these classes of tax, and the methodology developed provides insight into the nature and degree of the distortions provided by these features of the tax system.

## 1.1 Related Literature

The fact that a non-proportionate income tax may affect risky investment is not new to the literature. It is well-known that a tax disallowing losses, for example, preferentially favors gains over losses and generally distorts taxpayer behavior from what it would be if there were a proportionate tax, or no tax at all. Moreover, it is understood that any non-linearity in a tax system, such as a progressive rate structure, can impose a significant burden on risk taking. The work of Stiglitz (1969, 1972) and Sandmo (1985) are important theoretical studies of the effect of taxation on risk taking. The subsequent work of Kaplow (1991, 1994) introduces a general equilibrium methodology and establishes principles for determining equivalence between tax regimes. In addition, the work of Schenk (2000) and Zelenak (2006) recognizes that the usual arguments about the effect of proportionate taxes do not apply to non-proportionate taxes. Beyond the theoretical results and discussion, there is also an empirical literature, and the work of Gentry and Hubbard (2005) shows striking empirical evidence that entry into entrepreneurial activity by taxpayers is sensitive to convexity in the

[3]Under a convex tax, the tax on the average of two possible outcomes is less than the average of the tax reckoned on each outcome individually. This is the case, for example, under a progressive tax regime with greater marginal tax rates applicable to greater levels of income. A concave tax is the opposite, with the tax on the average being greater than the average of the taxes.

[4]Under a subadditive tax, the tax on a sum of incomes is less than the sum of the taxes levied on each of those incomes separately. A progressive tax on positive incomes with no offset for losses is an example of a subadditive tax. A superadditive tax satisfies the opposite inequality.

progressive rate schedule, with such convexity serving to discourage risk taking.

This paper moves beyond the existing literature by applying the general equilibrium techniques developed by Kaplow (1991, 1994) in the setting of non-proportionate taxes. This innovation is accomplished by extending the existing model to allow for trading in investment assets between the time of initial investment and final consumption, and by introducing techniques from financial economics to evaluate non-proportionate taxes as options on underlying risky assets. The key result of this approach is the identification of a certainty equivalent value for a non-proportionate tax on an investment portfolio. Such certainty equivalent values can be compared across investment strategies and quantify the degree to which a tax burdens risk taking of different types.

The results of this paper are also related to work in the corporate finance literature, where it has been long recognized that understanding of taxes on corporate income can be informed by the option pricing theory of financial economics. Green and Talmor (1985) and Majd and Myers (1986) present important research in this regard. The current paper is distinct from the corporate finance literature in that it employs the general equilibrium approach developed by Kaplow to analyze taxes in a context that takes into account fully the behavior of all parties, including the government. In addition, the current paper focusses on the question of the burden of a tax on individual investors instead of corporate investors. Nonetheless, there is significant connection between the corporate finance literature and the methodology set forth in this paper, and it is hoped that the two sets of literature may be able to benefit from and inform each other in the future.

## 1.2 Motivation in Terms of Financial Instruments

The approach taken by this paper and the methodology developed can be explained by interpreting tax burdens on investments in terms of financial instruments. Before turning to the details of the economic model involved, it is helpful to provide background and explain the motivation behind what follows.

As has been noted, the analysis of proportionate taxes dates back to the classic results of Domar and Musgrave (1944, 1945), and it has been broadly extended and placed in a general equilibrium setting.[5] The fundamental underlying idea of all this analysis can be expressed in the language of finance in terms of a forward contract on a risky asset and a riskless bond. For every unit of risky asset a taxpayer holds, his future obligation to pay taxes is the same as a short position in $\tau$ units of a forward contract on the asset, where $\tau$ is the constant

[5]See Kaplow (1991, 1994), who innovates and broadens the analysis by incorporating government action into the equilibrium and thereby obtains results that are truly of a general equilibrium character.

tax rate, coupled with a short riskless bond position under which he must pay in the future $\tau$ times the difference between the forward price and the current price of the asset.[6] The government, on the other hand, has a right to collect taxes paid that is the same as the long forward position and long riskless bond position complementary to the short positions of the taxpayer. The short forward position can be perfectly hedged by borrowing at the risk-free rate to invest in an offsetting position in the risky asset, and the long forward position can be perfectly hedged by the shorting an offsetting amount of the risky asset and investing at the risk-free rate. Both of these hedges are costless, and so the value of the forward contract is actually zero. Thus, the total cost of a proportionate tax is simply the same as that of a tax depending only on the riskless bond position, and hence depending only on the risk-free return to capital.

If a system of taxation is not proportionate, however, the usual analysis no longer applies since the obligation to pay tax and the right to collect tax are no longer the same as simple forward positions in the risky asset coupled with positions in a riskless bond. For example, if tax is levied on gains but no offsetting deduction is allowed for losses, then the taxpayer is short a call option on the risky asset while the government has the complementary long position in the same option. It is still possible to take positions in the risky asset that hedge the option position represented by the tax,[7] but the cost of hedging the option depends not only on the risk-free rate but also on the risk inherent in the underlying risky asset.[8]

A tax system can fail to be proportionate for a variety of reasons. As discussed in the preceding paragraph, it may not allow for loss-offsetting deductions. In fact, any progressive tax system also fails to be proportionate, since different levels of return are taxed at different rates. Moreover, a system that allows taxpayers discretion about the timing or rate of tax applicable to returns also fails to be proportionate. A common example of this is the

[6] A forward contract, or more simply forward, means a contract under which one party sells and the other party buys an asset at a specified price at a specified exercise time. The specified price is called the "forward price" and is required to be the future value that would be obtained if the current cost of the asset were invested at the risk-free rate of return until the specified exercise time.

[7] The hedging required will generally evolve dynamically. At each point in time, a particular position in the risky asset will hedge the option position, but the size of the required hedge will change over time depending upon how the asset value changes.

[8] The present value of the cost of hedging an option is the same as the price of the option. The techniques that can be used to determine this price rely on replicating the option dynamically using hedging positions in the asset and then evaluating the present value cost of such a dynamic hedging strategy. This cost generally depends on the amount of risk inherent in the underlying asset, as well as other factors. For example, if the returns of the risky asset are log-normally distributed, then the Black-Scholes formula can be applied to determine the option price, and if the risk-free rate is zero and the asset pays no dividends, an at-the-money call option costs $\frac{1}{\sqrt{2\pi}} S_0 \sigma \sqrt{t}$, where $S_0$ is the current price of the asset in dollars, $\sigma$ is the volatility (risk) of the stock, and $t$ is the time to expiration of the option. Since $1/\sqrt{2\pi} \approx 0.40$, this is approximately equal to 40% of the volatility of the stock in dollars over the term of the option. See Bodie et al. (2005) and Hull (2000) for further details and discussion of option hedging and pricing.

choice taxpayers have about when to realize gains and losses on capital asset – they may often choose, perhaps, to realize losses sooner and gains later. Another example is the ability of U.S. firms to decide when to keep earnings of a foreign subsidiary permanently reinvested abroad – they may often choose, perhaps, to permanently reinvest earnings in foreign jurisdictions with low tax rates so as to avoid additional tax upon repatriation of those earnings.

Each of the foregoing examples of non-proportionate taxes can be expressed as a deterministic function of the stochastic return on the risky assets in question. That is to say, even though risky asset returns are uncertain at the start, once the return is known, there is a definite and known way in which the tax on the return can be computed. This computation may involve choice on the part of the taxpayer, but if the taxpayer's preferences are known from the start, then the choice he will make given any particular levels of return is also known from the start.

If the tax on a risky asset is deterministic, then, conditioned upon knowing the asset return level, the tax payable can, in general, be hedged with a (possibly dynamic) position in the underlying asset. The tax can be thought of as a complex option on the risky asset, and the cost of this option can be determined by option pricing techniques that compute the cost of the hedging strategy in present value terms. This cost is referred to as the certainty equivalent value of the tax, and it quantifies, in present value terms, the burden imposed by the tax on the individual investor, and the value of the right of the government to collect the tax from the investor. This value will generally reflect the volatility, i.e., risk, of the underlying asset in a specific way, meaning that the manner in which a non-proportionate income tax burdens risky returns is understandable in a precise and quantifiable way.

In order to implement the ideas of the foregoing discussion in a rigorous way, it is necessary to have a conceptual framework that allows for the modeling of various systems of taxation. The starting place is the general equilibrium model introduced by Kaplow (1991, 1994), and this paper builds on that foundation. The existing model is extended to allow for continuous trading of investment assets between the time of initial investment and the time of final consumption. This extension allows option pricing techniques to be applied to determine precisely the above-described certainty equivalent value for a tax. In Section 2, the extension of the existing model is described in detail, certainty equivalent values are shown to capture the cost of a tax regime in a precise way, and the extended model is applied to recover well-known results in the case of a proportionate tax. In Section 3, the extended model is applied to investigate the burden imposed by a tax that is proportionate on gains but does not allow offsets for losses. It is shown that such a tax: burdens risky returns and encourages portfolio diversification targeted at risk minimization, without re-

gard to expected portfolio return; places substantial burdens on ownership of call and put options, with heavier burdens for options that are more out of the money, and hence more levered and riskier; and discourages synthetic division of ownership using put-call parity, and therefore also discourages debt financing of risky investments. In Section 4, the model is applied to other classes of tax, including convex and concave taxes and subadditive and superadditive taxes, and insights are gained into relevant aspects of current U.S. tax law. Section 5 concludes, and proofs of various results are contained in the Appendix.

# 2 Extending the General Equilibrium Approach

In this section, the general equilibrium approach pioneered by Kaplow (1991, 1994) is extended in two ways. First, the possibility of trading and rebalancing investment portfolios is allowed between the starting time, at which initial investments of capital are determined, and the ending time, at which tax is levied by the government and consumption of investment value occurs. Second, the possibility of a non-proportionate tax on returns to investment is introduced. The calculation of the certainty equivalent value of such a tax is described using option valuation techniques to construct a hedge for the tax liability that is updated dynamically over the course of time. Finally, the techniques developed are applied to recover the well-known equivalence results of Domar and Musgrave (1944, 1945) and Kaplow (1991, 1994) in the case of a proportionate tax regime.

## 2.1 The Basic Model and Multiple Time Steps

Following Kaplow (1991, 1994), the model is based on an economy beginning at time $t_0 = 0$ and ending at time $t_1 = 1$. At the first time point, individuals receive an amount to be invested, and not consumed, until time $t_1$.[9] At time $t_1$, the government levies a tax on investment returns and individuals consume the entirety of their after-tax investments. The government is also permitted to invest during the period from $t_0$ to $t_1$, and individuals and the government allocate their investment between a riskless asset and a risky asset.[10] Long and short positions can be taken by any investor.[11]

The basic model is extended by allowing for multiple times between $t_0$ and $t_1$ at which costless trading in investment assets by individual investors and the government can occur. The requirement still holds, however, that no consumption of investment assets is allowed until time $t_1$. Thus, the trading that is allowed only rebalances investment holdings and does not permit investors to consume investment wealth early. The addition of additional time steps allows investors to manage their portfolios actively, giving them a richer and more realistic set of opportunities than is possible if only a single initial portfolio allocation choice is made. In addition, it allows for the use of option valuation techniques necessary to

[9]The assumption that no consumption of investments occurs between times $t_0$ and $t_1$ can be relaxed, provided that consumption decisions are deterministic functions of stochastic investment returns. Including the possibility of consumption complicates the model without providing significant additional insight for purposes of this paper, however, and so consumption of investments is assumed not to occur until time $t_1$.

[10]The model can be extended in a relatively straightforward way to include multiple risky assets, but the discussion here is limited to just one risky asset to avoid too much complication. Section 3.4 provides some numerical examples in the case of two risky assets.

[11]Note that a short position in the riskless asset amounts to borrowing, and individuals and the government pay the same interest rate on borrowing in this model.

quantify the effects of non-proportionate taxes on investment earnings.

It is assumed that there are $n-1$ equally spaced times between $t_0$ and $t_1$, namely $t_{i/n} = i/n$, for $i = 1, \ldots, n-1$. At each time $t_{i/n}$ for $0 \leq i \leq n$, the value of the risky asset either increases by a factor $u_n$ or decreases by a factor $d_n$ between time $t_{i/n}$ and $t_{(i+1)/n}$, where

$$u_n = \exp\left(\mu/n + \sigma\sqrt{1/n}\right) \quad \text{and} \quad d_n = \exp\left(\mu/n - \sigma\sqrt{1/n}\right). \tag{1}$$

These values are chosen so that, in the limit as $n \to \infty$, if there is an equal likelihood of $u_n$ and $d_n$ at each time increment, the distribution of asset values at time $t_1 = 1$ is lognormally distributed such that underlying normal distribution has mean $\mu$ and standard deviation $\sigma$. If $A_0$ is the initial value of the risky asset, this specification for asset value evolution implies that there are $i+1$ possible values for the asset value $A_{i/n}$ at time $t_{i/n}$, namely $A_0 u_n^j d_n^{i-j}$ for $0 \leq j \leq i$.

In contrast to the risky asset, the change in size of the riskless asset from time $t_{i/n}$ and $t_{(i+1)/n}$ is constant. It is assumed that the rate of growth in each interval is $\exp(r/n)$ so that the value of the riskless asset at time $i$ is $B_{i/n} = \exp(ir/n)$.

The particular choice for restricting asset values to the discrete approximation of a lognormal distribution specified above allows for the valuation of options on the risky asset to be made with the binomial tree method discussed in Hull (2000, Chapter 9).[12] In addition, this discretization is useful for deriving the certainty equivalent of a tax regime in Section 2.2. However, little depends on the particular choice of discrete approximation as $n \to \infty$ and continuous portfolio rebalancing is permitted. At the end of Section 2.2, this limit is taken, and the discrete times approach is replaced with continuous time formulation.

## 2.2 Certainty Equivalent of a Tax Regime

Consider a tax regime in which the government imposes a tax on investment returns at time $t_1$. The tax regime is specified by a deterministic rule that reckons an amount of tax for each investor based upon the final value of his investments and the initial value of his investments. An example of such a regime is a proportionate tax that levies a constant-rate tax on gains and allows an offset for losses at the same constant rate. Another example is a non-proportionate tax that levies a constant-rate tax on gains but disallows any offset for losses. The goal of this section is to define a certainty equivalent value of the tax regime for

[12] For this method to be applied, "risk neutral" probabilities for $u_n$ and $d_n$ are used rather than equal probabilities. See Hull (2000) for details.

each investor. This amount quantifies the burden of the tax borne by an investor in a single present value figure, and the sum of these amounts over all investors quantifies the aggregate value of taxes to be paid to the government in present value terms.

The notion of the certainty equivalent value is based upon the concept of tax regime equivalence introduced in Kaplow (1991, 1994). The idea is to determine an up-front lump sum amount for each investor and a set of offsetting portfolio adjustments for each individual and the government with certain properties.[13] To wit, if the government foregoes a tax at time $t_1$ and instead levies a tax at time $t_0$ in the lump sum amounts and if the offsetting portfolio adjustments are made by all parties, then: (1) the after-tax investment portfolio is the same at time $t_1$ for each investor as it would have been under the original tax regime; (2) the final investment portfolio value for the government is equal to the amount of tax the government would have collected under the original tax regime; and (3) the portfolio adjustments are all offsetting so that the government and investors in the aggregate do not alter the net holdings of any asset at any time. If it is possible to determine up-front lump sum amounts and portfolio adjustments with these properties, then a state of general equilibrium under the original tax regime corresponds to a state of general equilibrium under the alternative tax system with portfolio adjustments. Indeed, the change from the original tax regime to the new system allows markets to clear and does not alter after-tax outcomes for investors or the government at time $t_1$. Because the objective functions maximized by investors and the government depend only on these final outcomes, a state of general equilibrium will be preserved by a change from the original regime to the new system. In this sense, the new system is equivalent to the original tax regime, and the up-front lump sum amounts represent the certainty equivalent value of tax for each investor.

It is now necessary to determine whether it is in fact possible to find up-front lump sum amounts and portfolio adjustments of the sort described in the preceding paragraph. Proceed first under the assumption that the binomial model described in Section 2.1 holds so that there are $i+1$ possible values for the risky asset at time $t_{i/n}$.[14] In this model, a final pre-tax target portfolio specification is a function $F$ that has a non-negative value for a portfolio assigned to each of the $n+1$ final possible states of the world, which correspond to the $n+1$ final possible values for the risky asset. It is a standard result that for any such function $F$, there is an up-front amount of money and a dynamic plan of investment that will lead deterministically to the final values specified in $F$.[15] The function $F$ can be thought of as

[13]The question of whether such amounts and such adjustments exists, and how to calculate them if they do exist, is discussed below.

[14]The more general case of continuous time is handled below.

[15]The idea is to work backwards with a methodology called "dynamic programming". The $n+1$ final values for the portfolio are specified at time $t_1$. From this it is possible to determine the $n$ values and

representing a complex option on the risky asset with payoff function $F$, and the up-front amount can be thought of as the price of the option. In a state of general equilibrium under the original tax regime, each investor has chosen a final pre-tax payoff function $F$ that has an option price equal to the investor's initial investment amount. The investor purchases this option and follows the dynamic investment strategy necessary to obtain deterministically the desired final pre-tax payoff specified by $F$.

Just as each investor specifies a target final pre-tax payoff function $F$, so the government has a final payoff function associated with the investor, denoted $T_F$, which results from the levying of tax at time $t_1$ on the values given by $F$. The government could obtain this same final payoff, however, by foregoing taxation at time $t_1$ and instead investing the option price of $T_F$ at time $t_0$ and following the dynamic investment strategy necessary to obtain deterministically the final payoff specified by $T_F$. Moreover, the government could obtain the option price of $T_F$ at time $t_0$ by levying an up-front lump sum tax on the investor. The investor would then be left with the option price of $F$ minus the option price of $T_F$ to invest at time $t_0$, but if he were to take this amount and invest it according to his original dynamic strategy, adjusted by positions to counter exactly the government's new investments, the final after-tax payoff to the investor would be the after-tax amount under the original tax regime, namely the after-tax payoff function $F - T_F$. The option value of $T_F$ is thus the certainty equivalent value of the original tax regime for the investor. If the government levies an up-front lump sum tax in this amount, and if appropriate portfolio adjustments are made, then a general equilibrium under the original tax regime will be transformed into a general equilibrium under this new system.

The foregoing derivation of certainty equivalent values may seem restrictive in that it relies on the particular choice of asset price evolution represented by the binomial model. Much more general results hold, however. What is necessary is that markets be complete in the sense that it is possible to construct a dynamic replicating portfolio of any desired final payoff function $F$. If asset prices follow a suitably well-behaved distribution, such as a lognormal distribution, then such dynamic replicating portfolios always exist in continuous-time trading. More complex cases, such as those involving stochastic volatility, for example, may require the addition of more assets beyond the basic risky asset in order to make the market complete. As long as the model is made expansive enough to have complete markets, however, the general approach described here works. Even if there are multiple

portfolio choices at time $t_{1-1/n}$ that necessarily give rise to the final portfolios at time $n+1$. Continuing to work backward, at time $t_0$ there is a single price that is the up-front amount of money required. The investment steps taken at each point constitute the required dynamic plan of investment. See Hull (2000, Chapter 9) for further details.

risky assets, the process of pricing a final payoff option through the procedure of valuing the cost of replicating portfolios works, and this pricing procedure can be used to determine the certainty equivalent values of a tax regime.

In what follows, it is assumed that continous-time dynamic replicating portfolios exist for any final payoff function $F$. It is also assumed that the tax payment due at time $t_1$ under the tax regime represented by $T_F$ is given by $T_F = T(F - P_0)$, for some function $T$. Thus, tax is a function of the difference between the final portfolio value and the initial portfolio value. Under these assumptions, the certainty equivalent value for the tax on an initial investment of $P_0$ resulting in a final payoff of $F$ under the tax regime represented by $T$ is

$$\mathrm{CEV}(P_0, F, T) = \mathrm{PV}(\mathbf{E}[T(F - P_0)]),$$

where the expectation is taken with respect to the risk-neutral distribution of values of $F$, and where PV denotes the present value function.[16] This is the continuous-time expression of the certainty equivalent value notion developed above for discrete-time situations.

## 2.3 Revisiting the Proportionate Tax

It is useful to see how the concept of certainty equivalent value developed above applies to a proportionate tax regime. Under such a regime, tax is levied at a constant rate $\tau$, and so the tax payment from an individual to the government at time $t_1$ is

$$T_{\mathrm{prop}}(P_1 - P_0) = \tau\ (P_1 - P_0), \tag{2}$$

where $P_i$ is the value of the individual's portfolio at time $t_i$. Note that it is possible for this payment to be negative, if $P_1 < P_0$, and in this case the government allows an offset of losses sustained.

For investor with an initial investment of $P_0$ and a final payoff function $F$, the certainty equivalent value of this tax regime is $\mathrm{CEV}(P_0, F, T_{\mathrm{prop}})$, which is the option price of the government's payoff function $T_{\mathrm{prop}}(F - P_0)$. Because of the structure of this tax, the government's payoff function is $T_{\mathrm{prop}}(F - P_0) = \tau(F - P_0)$, and so the option price of this payoff is simply $\tau$ times the difference between the option price of a final payoff of $F$ and the option price of a final payoff of $P_0$. The option price of a final payoff of $F$ is $P_0$, since this is the ini-

[16] Recall that this function consists simply of multiplication by $B_0/B_1$, the ratio of the current value of the risk-free asset to its future value.

tial amount invested to obtain the final payoff $F$, and the option price of the constant payoff $P_0$ is $(B_0/B_1)P_0$, which is just the present value of $P_0$.[17] Combining these findings, it follows that $\mathrm{CEV}(P_0, F, T_{\mathrm{prop}})$ is equal to $\tau P_0\,(1-(B_0/B_1))$. If one writes $B_0/B_1 = \exp(-r)$, the certainty equivalent value of the proportionate tax is thus seen to be $\tau P_0(1-\exp(-r))$, which is also equivalent to a value of $\tau P_0(\exp(r)-1)$ payable at time $t_1$. This is the well-known result of Domar and Musgrave (1944, 1945) and Kaplow (1991, 1994) that a proportionate tax is equivalent to a tax on the risk-free rate of return to initial investment value.

It is useful to express the foregoing result in the language of financial instruments as well. From this perspective, the government's tax payoff function is a combination of forward contract for a payoff in the amount $F$ at time $t_1$ and a riskless bond, namely

$$T_{\mathrm{prop}}(F-P_0) = \tau\ (F-P_0) = \tau\ \overbrace{(F-(B_1/B_0)P_0)}^{\text{Forward Payoff}}\ +\ \tau\ \overbrace{((B_1/B_0)P_0-P_0)}^{\text{Bond Payoff}}. \tag{3}$$

The forward contract has present value zero,[18] and the present value of the bond payoff is $\tau P_0\,(1-(B_0/B_1))$, and this is the value for $\mathrm{CEV}(P_0, F, T_{\mathrm{prop}})$ calculated above.

# 3 Applications To a Tax with No Loss Offsets

In this section, applications and results of the model in Section 2 are developed in the case of a tax that is proportionate for gains but that disallows loss offsets. Under this tax regime, if an individual makes an investment of $P_0$ dollars at time $t=0$ and his portfolio is worth $P_1$ dollars at time $t=1$, the end of his investment, then the tax payment due from the individual to the government at time $t=1$ is

$$T_{\mathrm{NL}}(P_1-P_0) = \tau\ \max(P_1-P_0, 0). \tag{4}$$

The approach is of this section is to consider specific numerical examples to gain insight into the effects of the $T_{\mathrm{NL}}$ tax. The findings are robust to a variety of choices for parameters underlying the numerical examples, and in fact theorems generalizing the results obtained can be proven in each case. The detailed statements and proofs of these theorems are

[17] Since the riskless asset has value $B_0$ at time $t_0$ and value $B_1$ at time $t_1$, the present value of a certain payoff at time $t_1$ is equal to $B_0/B_1$ times the amount of that certain payoff.

[18] The price of the forward contract is zero since the present option value of $F$ is $P_0$, the amount the investor initially has available to invest, and the present value of a certain payoff of $(B_1/B_0)P_0$ is also $P_0$. Thus the option value of the difference of these two sets of payoffs is zero.

included in the Appendix rather than in the current section so as to allow for greater clarity of exposition.

In Section 3.1, the burden of the $T_{\text{NL}}$ tax is shown to be greater for assets with more risk. It is also demonstrated that this tax actually burdens the expected level of return in the case of investment in the market portfolio when the Capital Asset Pricing Model describes returns, although this is not generally true for multi-factor market models or other investment portfolios. In Section 3.2, riskless debt is shown to be subject to a lower burden than that for any risky asset under the $T_{\text{NL}}$ tax, although some portfolios with a small portion of risky investment and the rest riskless are seen to have approximately the same burden as a riskless portfolio. In Section 3.3, the high tax burden of options is demonstrated, and it is further shown that synthetic division of asset ownership using put-call parity, and debt financing in particular, is subject to a greater aggregate burden for taxpayers under the $T_{\text{NL}}$ tax than direct proportionate ownership of the underlying risky asset. Finally, in Section 3.4, the situation of two risky assets is considered, and it is demonstrated that diversification of portfolio holdings across risky assets is generally favored. The result of this final section is similar to previous findings, but the innovation is the simultaneous modeling and analysis of two risky assets at once.

## 3.1 Burden on Risky Assets

In this section, the burden of the $T_{\text{NL}}$ tax on the risk level of an investment is studied, and investor incentives to minimize risk are examined. Consider an investment of $P_0$ dollars, in a risky asset $A$ with initial value $A_0$ and final payoff $A_1$. The certainty equivalent value of the $T_{\text{NL}}$ tax for this investment is

$$\begin{aligned}\text{CEV}(P_0, A_1, T_{\text{NL}}) &= \text{PV}(\mathbf{E}[T_{\text{prop}}(P_0(A_1/A_0 - 1))]) \\ &= \tau P_0 \text{PV}(\mathbf{E}[\max(0, A_1 - A_0)])/A_0.\end{aligned}$$

The present value of the expectation in this final expression is exactly the price of an at-the-money call on the risky asset $A$, and if $A_1$ has a lognormal distribution with underlying volatility $\sigma$, this value can be calculated using the Black-Scholes formula to find[19]

$$\text{CEV}(P_0, A_1, T_{\text{NL}}) = \frac{\tau P_0 \sigma}{\sqrt{2\pi}}.$$

[19]See footnote 8 for more information about this calculation, and note that the time to expiration of this call is $t = 1$.

From this expression, it is clear that the burden of the tax on this investment increases linearly with the volatility, $\sigma$, of the asset. Figure 1 shows this relationship graphically in the case of a tax rate of $\tau = 35\%$.

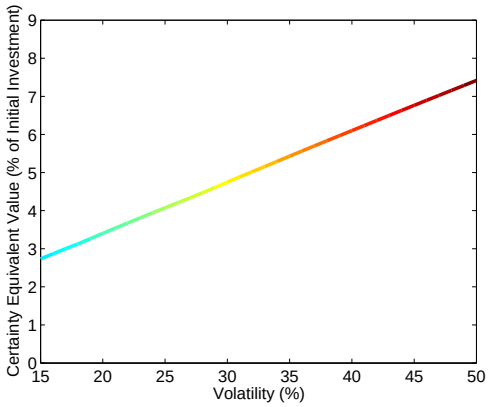



Figure 1: The figure shows how the tax cost of a position invested 100% in the risky asset varies with the volatility, $\sigma$, of the asset.

The calculations of the previous paragraph show that the $T_{\text{NL}}$ tax favors investments in lower volatility assets. If an investor is choosing between a single asset with significant idiosyncratic risk and a diversified mix of assets with little idiosyncratic risk, the $T_{\text{NL}}$ tax will thus weigh in favor of diversification. Moreover, if a single-factor model for asset returns, such as the CAPM, holds, then the greatest diversification, and lowest tax burden, is achieved by holding the market portfolio. If such a model holds, and if $A$ represents an investment in the market portfolio, then the tax burden is equal to

$$\text{CEV}(P_0, A_1, T_{\text{NL}}) = \frac{\tau P_0(\mu_{\text{mkt}} - r)}{S_{\text{mkt}}\sqrt{2\pi}},$$

where $\mu_{\text{mkt}} - r$ is the excess expected return on the market portfolio over the risk-free rate, and where $S_{\text{mkt}}$ is the Sharpe ratio[20] for the market portfolio. Historically, the value of $S_{\text{mkt}}$ has been approximately 0.4,[21] and so $S_{\text{mkt}}\sqrt{2\pi} \approx 1$. Substituting this value, it is seen that

$$\text{CEV}(P_0, A_1, T_{\text{NL}}) \approx \tau P_0(\mu_{\text{mkt}} - r),$$

[20]The Sharpe ratio is defined as the reward-to-variability ratio of returns, which in this case is the ratio of the excess market return to the standard deviation of market returns.

[21]This number is based on annualized excess return and volatility figures for the market portfolio from Professor Kenneth French's website for the years 1926 to 2009.

and this means that the tax burden placed on the market portfolio is approximately equal to the nominal tax rate on gains, $\tau$, times the expected excess market return. This stands in sharp contrast to the well-known result from Section 2.3 that proportionate taxes only burden the risk-free return to initial invested capital.

The foregoing results demonstrate that the $T_{\mathrm{NL}}$ tax encourages diversification and may effectively burden expected risky returns. Such a burden on expected risky returns may not be desirable, however, since it may prevent optimal levels of investment in risky assets, as opposed to riskless assets. The question of mixed portfolios of risky and riskless assets is taken up in the next section. It should also be noted that encouragement of diversification may not always lead to optimal results. To see why this is so, suppose that returns in the market are driven by multiple factors, rather than just the single factor of the CAPM. In this situation, in the absence of taxes, an investor will choose investment weights across the factors by taking into account not only the risks of the factors, but also their expected levels of return. The tax burden, however, will depend only upon the risk of the overall portfolio the investor selects and not upon its expected return. As a result, taxes may distort an optimal allocation of capital by the investor, causing him to target a less risky portfolio than would be optimal.

## 3.2 Mixing Risky and Riskless Assets

In this section, the burden of the $T_{\mathrm{NL}}$ tax on an investment portfolio with a mixture of a risky and a riskless asset is analyzed. Consider two simple strategies that can be employed by an investor. The first strategy, denoted $B$ is one in which the investor puts all of his initial investment in the riskless asset and leaves it there until time $t_1$. The second strategy, denoted $A$ is one in which the investor puts all of his initial investment in the risky asset and leaves it there until time $t_1$. Much more complex strategies are of course possible, but it is informative to start with these basic choices. Given parameters for the asset returns and the rate of tax on gains, it is possible to use the methods of Section 2 to compute the certainty equivalent value of the tax on each of these strategies. Assume that

$$\mu = 11.0\%, \quad \sigma = 18.9\%, \quad r = 3.6\%, \quad \text{and} \quad \tau = 35\%, \tag{5}$$

where $\mu$ and $\sigma$ are the quantities appearing in (1) that describe the lognormal distribution followed by the risky asset's value, and where $r$ is the continuously compounded rate of

return on the riskless asset.[22] Under these assumptions, the certainty equivalent values of the two simple strategies are [23]

$$\mathrm{CEV}(P_0, B, T_{\mathrm{NL}}) = 1.24\%P_0 \quad \text{and} \quad \mathrm{CEV}(P_0, A, T_{\mathrm{NL}}) = 3.25\%P_0.$$

It is notable that the certainty equivalent value for the risky asset strategy is more than twice the value for the riskless asset strategy. As was seen in Section 3.1, the tax burden on an investment in an asset increases with volatility, and since the $B$ strategy has no volatility, it has the lowest burden available.

Moving beyond the two simplest strategies, consider a mixed strategy in which an investor places a fraction $w$ of his initial investment value in the $A$ strategy and the remaining fraction $1-w$ in the $B$ strategy. The investor does not trade any investments over the course of time from $t_0$ through $t_1$. Figure 2 illustrates how the certainty equivalent value of such a strategy varies with the choice of weight $w$. When $w = 0\%$ and $w = 100\%$, the certainty equivalent values calculated above for the $B$ strategy and the $A$ strategy are recovered, respectively. In between, however, it is notable that the certainty equivalent value remains roughly constant for small values of $w$. This corresponds to the fact that, for small values of $w$, potential losses in the risky asset are generally outweighed by deterministic gains in the riskless asset. As long as the net change in value of the portfolio over time is positive, the non-proportionate tax is the same as a proportionate tax, and the proportionate tax yields the same certainty equivalent value for any choice of $w$. Thus, the non-proportionate tax does not discourage all risk taking equally, but rather it burdens most heavily those portfolios that are most likely to lead to net portfolio losses. The non-proportionate tax therefore treats preferentially portfolios that are hedged in such a way as to make losses less likely.

[22] These numbers are annualized statistics based on monthly market data available on Professor Kenneth French's website for the period from 1926 through 2009.

[23] The calculation of these amounts was accomplished using the binomial option pricing model described in Section 2 with $n = 500$ time increments. In general, this is the methodology used for calculating all certainty equivalent values in this Section 3.2 and in Section 3.3.

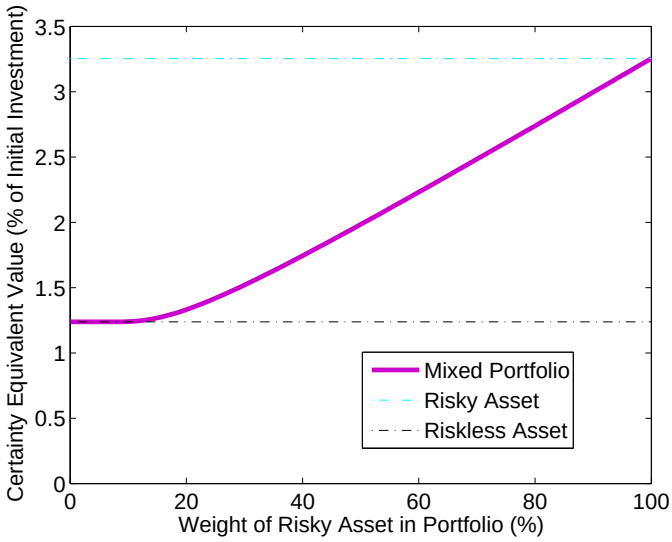


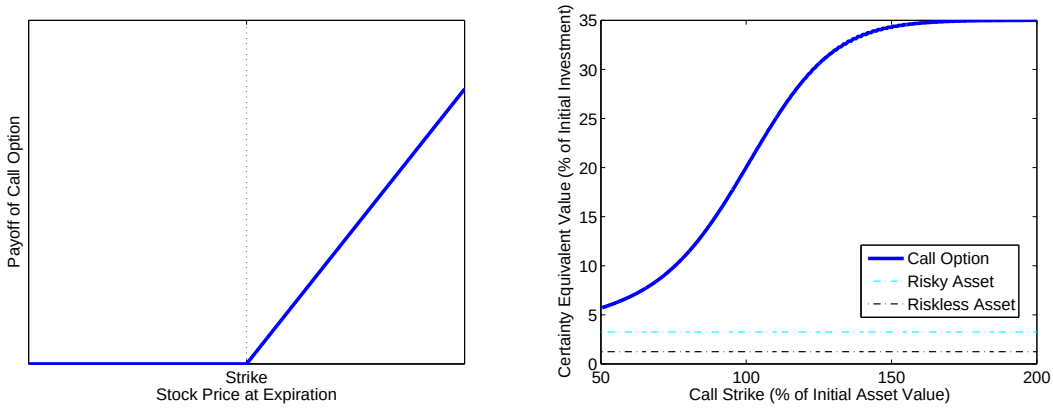

Figure 2: The figure illustrates the certainty equivalent value for the non-proportionate tax $T_{\text{NL}}$ in the case of a portfolio constructed to have an initial weight, $w$, invested in the risky asset and a constant complementary weight, $1 - w$, invested in the riskless asset at time $t_0$, with no further trading in assets occurring from time $t_0$ through time $t_1$. The weight varies from 0% to 100% along the horizontal axis, and the certainty equivalent value is expressed as a percentage of the initial investment value, $P_0$.

## 3.3 Analysis of Options and Put-Call Parity

Moving beyond combinations of the simple $A$ and $B$ strategies, it is interesting to investigate how the non-proportionate tax described in (4) burdens the portfolios of an investor who targets a final payoff function $F$ that depends on the risky asset in more complex ways. To this end, the burden on final payoffs functions in the form of the payoffs for put and call options are now considered.

Figure 3 illustrates a typical payoff function for a call option (on the left) and the certainty equivalent value for a portfolio with final payoff function equal to a call option on the risky asset (on the right). The horizontal axis in the second graph indicates the strike, as a percentage of the initial value of the risky asset, of the option. As the strike price decreases, the payoff on the call option is closer to the payoff on the underlying risky asset, and ultimately a call option with a strike price of zero is the same as an interest in the underlying risky asset. Accordingly, the certainty equivalent value of a portfolio of call options tends to that of the $A$ portfolio described in Section 3.2 as the strike price tends toward zero. As the strike price increases, however, the certainty equivalent value of the call option becomes substantially larger and in fact tends to $\tau = 35\%$.[24]

[24]Appendix A.4 contains a proof that the certainty equivalent value of the $T_{\text{NL}}$ tax on the portfolio of




(a) Payoff Function for a Typical Call

(b) Tax Cost of a Call

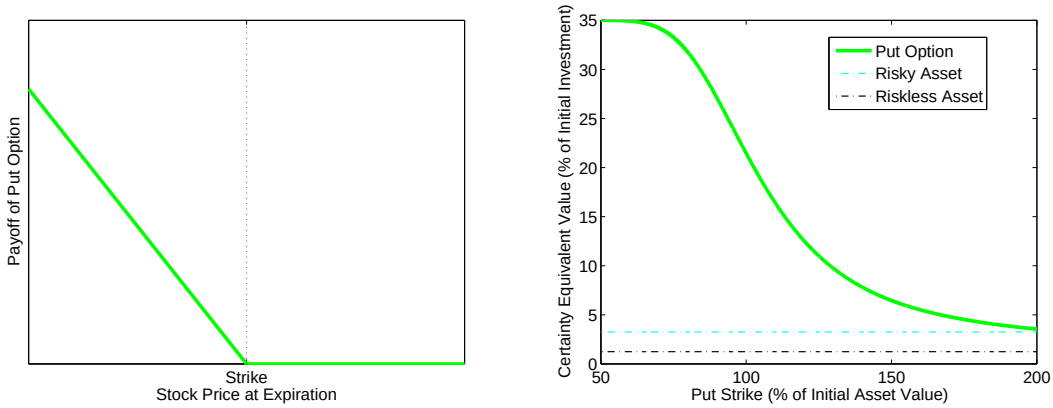
Figure 3: The figure on the left shows a typical payoff function for a call option, with nothing paid if the risky asset value is not greater than the strike price, and with the difference between the risky asset value and the strike price received otherwise. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function $F = C_1(K)$ equal to a call option payoff on the risky asset. The strike, $K$, of the call option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

The burden on put options is similar to that described for call options, except that the burden is greater for lower strike values than higher strike values. Figure 4 illustrates a typical payoff diagram for a put option (on the left) and the certainty equivalent value for a portfolio with final payoff function equal to a put option on the risky asset (on the right).[25] The certainty equivalent value for a put option tends to $\tau = 35\%$ of the invested capital as the strike price tends to 0. For large strike prices, the certainty equivalent value for a put option portfolio tends to that for a portfolio invested exclusively in the risky asset. In this way, the burden on the put option is much like a mirror image of the burden on a call option.

It is a natural question to ask whether a short option position might produce a result different from a long position, and so the case of a short put option is now investigated. Suppose that an investor selects a final payoff function $F$ with the property that $F$ has the same payoff as a certain number of short put positions and a long position in the riskless asset in an amount that has a final payment equal to the strike price of the put times the number

options increases with the strike price. The same section also contains proofs about the limiting value of the value as $K$ tends toward 0 and $\infty$.

[25] Appendix A.4 contains proofs describing the behavior of the certainty equivalent value for a portfolio of put options.




(a) Payoff Function for a Typical Put

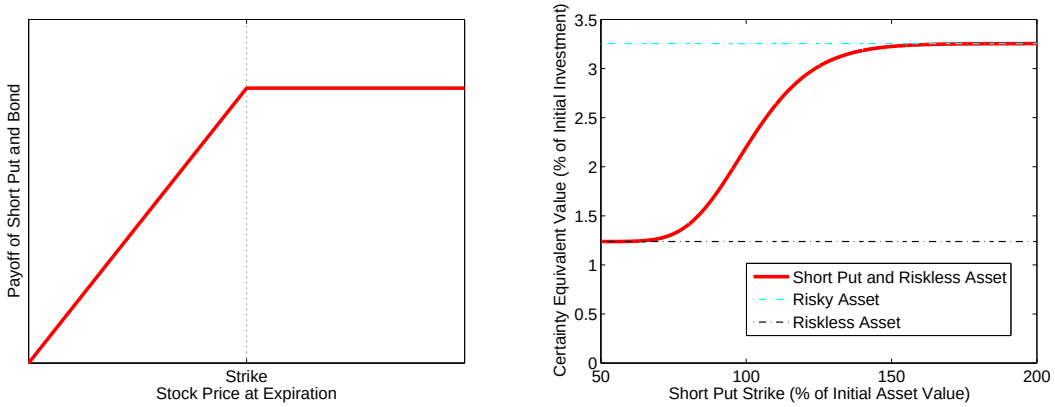
(b) Tax Cost of a Put

Figure 4: The figure on the left shows a typical payoff function for a put option, with nothing paid if the risky asset value is not less than the strike price, and with the difference between the strike price and the risky asset value received otherwise. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function $F = P_1(K)$ equal to a put option payoff on the risky asset. The strike, $K$, of the put option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

of short put positions.[26] This strategy is designed so that the long bond positions prevent the investment portfolio from ending with a negative total amount for any final value of the risky asset. The typical payoff diagram for such a combination of short puts and riskless assets is shown on the left in Figure 5. On the right, the figure shows the certainty equivalent value for this portfolio, with the strike varying along the horizontal axis. The figure indicates that the burden on this portfolio increases with increasing strike, the opposite of what happens for the long put portfolio. Moreover, the burden grows from a level equal to that of the Bond strategy to a level equal to that of the Stock strategy. Thus the short put position, which is effectively a "short volatility" strategy, allows an investor to achieve a tax burden below that of a position invested 100% in the risky asset, but it is never lower than the burden on a position invested 100% in the riskless asset.

The foregoing results can be combined to determine what the tax burden is on a synthetic division of ownership of the risky asset, $A$, among taxpayers who use the put-call parity

[26]The specific number of positions taken is the number that can be financed by the initial investment amount, $P_0$, available to the investor. Thus, the number is chosen so that the entire portfolio consists of such positions.




(a) Payoff Function for a Typical Short Put and Bond

(b) Tax Cost of a Short Put and Bond

Figure 5: The figure on the left shows a typical payoff function for a short put option combined with a long position in the riskless asset in an amount equal to the strike of the put. The payoff is the same as that of the risky asset up to the level of the strike, and thereafter the payoff is limited to the amount of the strike. The figure on the right shows the tax cost, measured as the certainty equivalent value, for an investment portfolio with final payoff function $F$ of the type illustrated in the figure on the left. The strike of the put option varies along the horizontal axis and is expressed as a percentage of the current value of the risky asset. The certainty equivalent value is expressed as a percentage of initial investment value.

relationship, namely,

$$\text{(Risky Asset)} \;=\; \text{Call} \;+\; \text{(Riskless Asset)} \;-\; \text{Put}.$$

In this equation, the put and call have the same strike price, and amount of the riskless asset is chosen so that the final payoff amount of this asset is equal to the common strike price of the put and the call. The payoff of the risky asset is exactly equal to the combined payoff of the positions on the right hand side of this equation, and so an investment in the right hand portfolio is the same as an investment in the risky asset.

Consider two alternative scenarios. In the first, there is a single investor with an investment of $P_0$, his entire portfolio, in the risky asset. Assuming the parameter values specified in (5), the certainty equivalent value of the tax burden on each investor is 3.25% of the initial investment value. In the second scenario, there are two investors, one of whom has invested his entire portfolio in call options on the risky asset with a strike price of 100,[27] and the other of whom has invested her entire portfolio in short put options on the risky asset with a strike price of 100 and long riskless asset positions that have a final payoff amount equal to 100. The initial wealth of the call investor is $9.30\%P_0$ and the wealth of the short put option investor is $90.70\%P_0$. Thus their aggregate initial wealth is the same as that of the single investor in the first scenario, and the relative sizes of their wealth are chosen so that their positions combine to represent an aggregate initial ownership of exactly $P_0$ dollars worth of the risky asset.[28] Because the aggregate pre-tax position of the two investors is the same as that of the single investor, it might seem that the aggregate certainty equivalent value of the tax should not change. However, the investor who holds call positions no has a certainty equivalent value of 20.02% of the initial investment value and the investor who holds short put positions has a certainty equivalent value of 2.20% of the initial investment value. In weighted average terms across the entire group, the aggregate certainty equivalent value is now

$$20.02\% \left(\frac{9.30}{100}\right) + 2.20\% \left(\frac{90.70}{100}\right) = 3.86\%.$$

Thus the overall tax burden on the group has increased from 3.25% to 3.86%, an increase of

[27] For purposes of this example, assume that the initial value of one unit of the risky asset is 100.

[28] These numbers are computed using the assumed parameters of (5) since the price of a call option struck at 100 is 9.30, and the price of a short put option plus a long riskless asset position of the type described is 90.70. As before, it is assumed that the initial price of the risky asset is 100.

nearly 20%.

The result of the last paragraph is typical of the situation when put-call parity is used to reorganize the division of ownership across a group with options under the $T_{\text{NL}}$ tax.[29] There is generally an aggregate increased tax burden in terms of the aggregate certainty equivalent value. The increased burden is not spread evenly, however. As the foregoing example illustrates, some individuals may have a much higher burden (as the call holders did), and some a significantly lower burden (as the put holders did).

This result is of particular interest because a division of ownership using debt financing is essentially the same as the type of division based on put-call parity described the above example. The equity holders have a call option on the underlying risky asset, and the debt holders have the remaining short put and long riskless asset positions. The above analysis thus shows that, in this non-proportionate tax regime, debt financing results in a larger aggregate tax burden, as measured by the certainty equivalent value, than does 100% equity ownership.

Figure 6 illustrates the results of the example above for various alternative values of the strike value for the options involved. As can be seen from the figure, the largest increase in tax burden coming from the synthetic strategy occurs for strikes close to 100% of the initial value of the risky asset.

[29]This results from the fact that the $T_{\text{NL}}$ is subadditive. See the proof of this result in Appendix A.3.

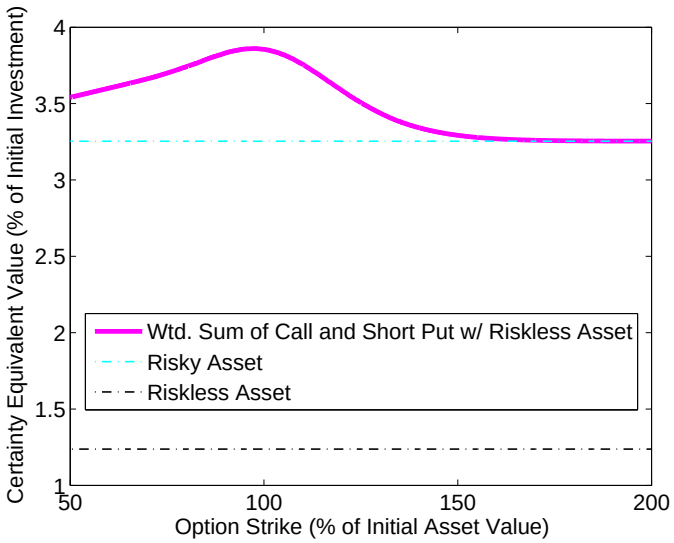



Figure 6: The figure shows the tax burden, measured as the average certainty equivalent value, for the group of investors described under the second scenario of the put-call parity example of Section 3.3. Some of these investors hold call options on the risky asset, and some have short put options with the same strike as the calls and long riskless asset positions with final payoffs equal to the strike of the calls. The relative number of each type of investor is chosen so that, in the aggregate, the pretax position of the group is equivalent to ownership of the risky asset. The strike value used for the calculation varies along the horizontal axis. The certainty equivalent value is expressed as a percentage of initial investment value.

## 3.4 The Case of Two Risky Assets

As a final numerical example, consider the case of two risky assets. Until now, the focus of the analysis has been on only one risky asset and one riskless asset. However, it is possible to add a second risky asset to the set of available investments and to calculate certainty equivalent values for tax burdens on the possible investment portfolios. Option values can no longer be calculated using the simple binary tree employed in the previous sections. Other techniques are available, however, and the computations performed below use the Monte Carlo simulation techniques described in Hull (2000, Chapter 16).

It is assumed for purposes of the numerical examples in this section that each of the two risky assets has a lognormal distribution and that the mean and standard deviation of the underlying normal distribution for both assets are $\mu = 11.0\%$ and $\sigma = 18.9\%$, respectively. Also, the continuously compounded rate of return on the riskless asset is $r = 3.6\%$, and the constant tax rate on gains is $\tau = 35\%$. The degree of correlation between the returns of the two risky assets is varied in the examples that follow.

Suppose first that the returns on the two risky assets are uncorrelated and that an investor chooses a positive fraction of his initial investment value to invest in each of the three available assets. Suppose further that the investor does not change investment holdings from time $t_0$ through time $t_1$. Each possible portfolio of this type is completely described by the initial investment choice, namely the aggregate fraction of the initial value invested in the two risky assets (with the balance being invested in the riskless asset) and the fraction of the risky amount allocated to the second risky asset. Figure 7 illustrates the certainty equivalent values for such portfolios. It is evident from the figure that riskless portfolios tend to have the lowest tax burden, while the highest burden is obtained for portfolios that hold exclusively just one of the risky assets. For any given level of aggregate investment in risky assets, the minimum tax burden is obtained at an equal division of holdings between the two risky assets.

If the risky assets have a non-zero correlation, the same computations can be performed. The figure on the left in Figure 8 illustrates the case in which the correlation is 50%, and the figure on the right illustrates the case in which the correlation is $-50\%$. In both cases, it remains true that the highest tax burden is borne by portfolios concentrated in a single risky asset. Also, the lowest certainty equivalent value for any particular level of aggregate risky investment is achieved when a portfolio is evenly divided between the two risky assets.[30] In addition, examination of the same portfolio weightings at different levels of correlation

[30]The fact that an even division minimizes the certainty equivalent value in all cases considered here is a result of the fact that the assets chosen for this numerical example have the same underlying standard deviation for their returns.

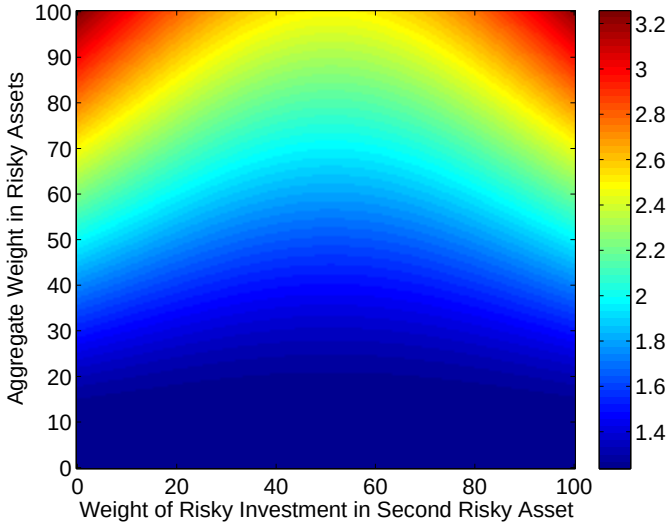
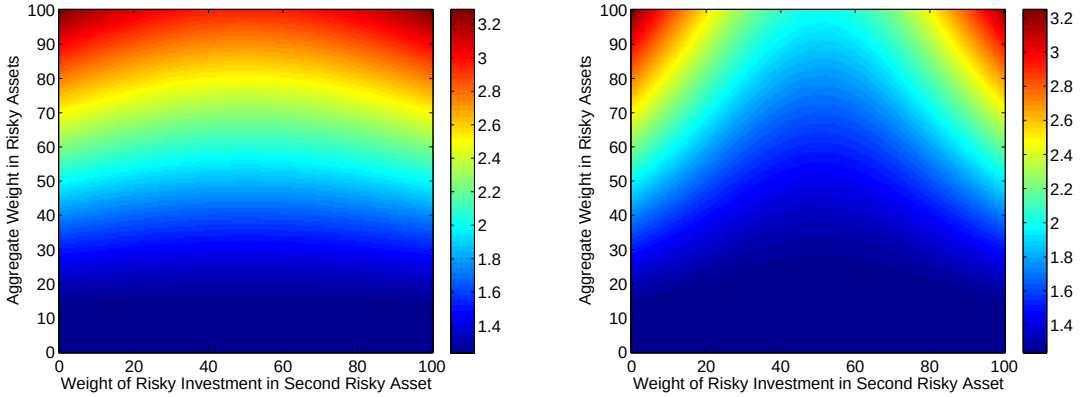



Figure 7: The figure shows the tax burden, as measured by the certainty equivalent value, on portfolios that contain a mix of two uncorrelated risky assets and a riskless asset. The aggregate weight of the portfolio invested in the two risky assets is indicated along the vertical axis. The weight of the fraction of the risky portion of the portfolio that is allocated to the second risky asset is indicated along the horizontal axis. The colors represent the certainty equivalent value of the tax on the portfolio as a percentage of initial asset value.

shows that more negative correlations lead to lower certainty equivalent values, and more positive correlations lead to higher certainty equivalent values. In general, the burden of the non-proportionate tax falls more heavily on less diversified portfolios.




(a) Correlation of 50%

(b) Correlation of −50%

Figure 8: The figure on the left shows the tax burden, as measured by the certainty equivalent value, on portfolios that contain a mix of two risky assets, with a correlation of 50%, and a riskless asset. The aggregate weight of the portfolio invested in the two risky assets is indicated along the vertical axis. The weight of the fraction of the risky portion of the portfolio that is allocated to the second risky asset is indicated along the horizontal axis. The figure on the right is the same except that the correlation between the two risky assets is −50%. In both figures, the colors represent the certainty equivalent value of the tax on the portfolio as a percentage of initial asset value.

# 4 Other Non-Proportionate Taxes and U.S. Tax Rules

This section considers classes of non-proportionate taxes more general than just $T_{\text{NL}}$ and in particular examines some related features of current U.S. tax rules that may burden or incentivize risk taking and division of ownership. Section 4.1 investigates properties of convex and concave taxes. Section 4.2 analyzes features of subadditive and superadditive taxes. In both sections, some preliminary policy considerations stemming from the paper's framework are set forth.

## 4.1 Convex and Concave Taxes

A tax function $T$ is said to be convex if

$$T(\alpha x + (1-\alpha)y) \leq \alpha T(x) + (1-\alpha)T(y), \tag{6}$$

for all values of $x$ and $y$ and for all $0 \leq \alpha \leq 1$. For any convex tax, the riskless asset is always less heavily burdened by the tax than any risky asset.[31] A tax function is said to be concave rather than convex if the opposite inequality from the one in (6) holds. For such taxes, risky returns are burdened less heavily than riskless returns, and so the tax effectively provides a subsidy for risk taking.[32] Finally, a tax function is said to be flat if exact equality holds in (6), and in such a case, the tax is simply proportionate, so that it neither burdens nor subsidizes risk taking.[33]

The U.S. tax system is not entirely convex, concave or proportionate, but it does exhibit aspects of each of these types of regimes. A progressive tax is convex, and since principles of progressivity are reflected in much of the U.S. tax system, the system has significant convexity. However, there are important ways in which the U.S. tax system is not convex, particularly in the case of capital gains. Under current law, the tax rate on net capital gains is 0% for low bracket taxpayer and 15% for those in higher brackets.[34] However, net capital losses may be taken, up to the extent of $3,000,[35] against ordinary income, and thus qualify for a marginal rate of tax greater than zero for the low bracket taxpayers and greater than 15% for the high bracket taxpayers. As a result, when net capital gains are reduced to the point that they change to net capital losses, the marginal rate of the taxpayer increases. After $3,000 of net capital losses have been taken for a year, the marginal rate goes down to zero for any further losses.[36] The combined result is that the tax on capital gains has aspects of convexity, concavity, and proportionality. For taxpayers with relatively modest incomes and relatively small exposures to the possibility of gain or loss, concavity dominates, and the tax regime provides an incentive to take risk. For taxpayers with a mixed possibility of either a substantial gain or a substantial loss, convexity dominates, and the tax regime penalizes risk taking. Finally, for taxpayers with primarily a substantial gain risk or a substantial loss risk, but not both, proportionality dominates, and the tax regime neither incentivizes nor penalizes risk taking.

The structure of the current capital gains rates is somewhat problematic. Those with the smallest incomes and investments are given the most incentive to take risk by the tax

[31] This is observed in Section 3.2 to be the case for the convex tax $T_{\text{NL}}$. It is proven in general in Section A.2 of the Appendix.

[32] The proof of this fact uses Jensen's inequality in the same way that the proof of Theorem A.2 does, except that the inequality is reversed because the tax is concave.

[33] This follows because proportionate taxes levy the same burden on all initially invested capital, regardless of investment choices. See Section 2.3 for more details.

[34] Section 1(h) of the Internal Revenue Code sets forth the rate of tax on capital gains for individuals.

[35] Section 1211(b) of the Internal Revenue Code disallows net capital losses beyond $3,000 for an individual in any one year.

[36] Additional losses may be carried forward to future years. For purposes of the present discussion, however, a one-year period is assumed.

system, while those with much larger investments may be penalized for taking risk. Arguably neither incentive nor penalty is appropriate, since both distort optimal behavior. Even if some distortion is desired, however, there seems to be little policy reason for encouraging risk-taking in those with less to invest.

A re-working of the rate structure for capital gains seems sensible, and the framework and tools developed in this paper can help give guidance regarding considerations of convexity and concavity that may often be overlooked or unappreciated as the true sources of burden and incentive for risk-taking. Ideally, a completely proportionate tax could be adopted so as not to distort optimal choices. However, even if some non-proportionality is desired, on either grounds of fairness[37] or administrability,[38] at least the methodology presented here can help guide the precise structuring so as not to result in unintended consequences. For example, if convexity is desired overall, it should be maintained uniformly throughout all levels of gain and loss, and in particular there should not be any losses that enjoy a marginal rate of tax higher than that on gains.

## 4.2 Subadditive and Superadditive Taxes

A tax function $T$ is said to be subadditive if

$$T(x+y) \leq T(x) + T(y), \tag{7}$$

for all values of $x$ and $y$. If a tax is subadditive, then unified ownership by a single taxpayer is preferred to divided ownership by multiple taxpayers. This was seen in Section 3.3 to be the case for debt financing in the case of $T_{\text{NL}}$, a subadditive tax, and it is true more generally for any division of ownership.[39] A tax function is said to be superadditive if it satisfies the opposite inequality from the one in (7). Superadditive tax functions provide an incentive for divided ownership over unified ownership.

[37] Notwithstanding the analysis of this paper about the true burdens of taxes on investments, it may appear more fair to tax those with lower incomes at lower rates on their capital gains. This could be so because of a misapprehension of the fact that it is convexity in tax, rather than absolute rate, that creates a burden on risky investment. It could also be so because of the effect of transaction costs or other frictions that have not been incorporated into the current model.

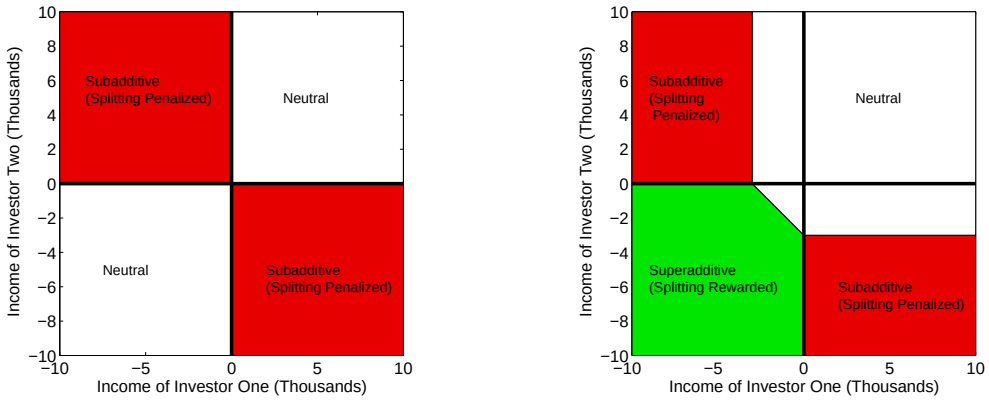
[38] It may not be deemed reasonable, as a practical matter to give, for example, full offsets for losses rather than having a $3,000 limit. There might be concern that if such a change were made, taxpayers would find ways to generate losses beyond those they should rightly be allowed to claim.

[39] Note that this result is closely related to issues regarding the marriage penalty and marriage bonus. Given a fixed progressive rate schedule, it is advantageous for two people to be able to split income and be taxed separately rather than be taxed together. This result holds because a progressive rate schedule is generally subadditive.

A tax that is proportionate or progressive for positive incomes and does not allow any offset for losses is entirely subadditive, and as such, it penalizes division of ownership. If such a tax is altered so that it allows for loss offsets up to a capped amount, however, then it becomes superadditive for certain values of $x$ and $y$. In particular, if a division of ownership between two investors results in losses for each beyond the capped amount, the two investors receive more favorable tax treatment than a single unified owner would. Figure 9 illustrates the ranges of pairs of earnings for two investors for which $T_{\text{NL}}$ is subadditive (left sub-figure) and for which $T_{\text{NL}}$ modified by a \$3,000 loss offset is subadditive and superadditive (right sub-figure).

The U.S. tax system generally allows for a capped amount of loss offsets,[40] and as such it encourages division of ownership whenever the individual pieces of the division result in losses. When the pieces present a mixed bag, however, with one losing significantly and one gaining significantly, unified ownership is more favored than divided ownership. It does not seem that this set of incentives is particularly desirable, and so this is another area in which revision of rules may be appropriate. An entirely proportionate tax would eliminate any incentives to divide or to unify ownership, but, as was noted above, such a tax may not be feasible on the grounds of either fairness or administrability. Failing such a sweeping change, however, it may still possible to relieve some of the unintended consequences for investor incentives by increasing the cap on capital losses allowed. The exact degree to which it should be increased is an empirical question that depends upon the realizations that investors have and the extent to which investment results fall into the red shaded area of Figure 9(b). The considerations set forth here, however, can help inform the analysis of empirical data to determine where the cap might be optimally set.

[40]For purposes of this discussion, only one period is studied, and so the ability to carry losses to another year is ignored.




(a) No Loss Offsets

(b) Loss Offsets up to $3,000

Figure 9: The figure on the left shows the regions in which the $T_{\text{NL}}$ tax burdens a division of ownership. The horizontal and vertical axes represent possible income outcomes for two investors who each own a portion of a single asset; together they own the entire asset. If a pair of income outcomes is in the red area, the division of ownership is more heavily burdened by the $T_{\text{NL}}$ tax than unified ownership by a single taxpayer would be. If it is in the white area, there is no tax advantage or disadvantage to division of ownership. The figure on the right shows the similar picture for a tax that is proportionate except that it allows $3,000 of loss offsets. In this case, the red and the white areas play the same role as in the figure on the left. There is now also a green area, however, in which division of ownership is tax favored relative to unified ownership.

# 5 Conclusion

The framework developed herein extends the general equilibrium techniques of Kaplow (1991, 1994) by incorporating trading in investor asset portfolios and applying option pricing techniques to evaluate the burden of non-proportionate taxes on investors in terms of certainty equivalent values. An important and immediate consequence is that non-proportionate taxes burden the risk in asset returns, but not the expected levels of return. If expected return levels are determined as functions of risk, then they may be burdened indirectly, but the existence of such a functional relationship depends on particular choices of market model and portfolio, such as an investment in the market portfolio under the Capital Asset Pricing Model. The finding that non-proportionate taxes burden the risky component of asset returns stand in sharp contrast to the well-known result that a proportionate tax burdens only the risk-free return to initial capital.

The new theoretical approach to analyzing and understanding tax burdens is applied to the convenient example of a tax that is proportionate for gains but does not allow offsets for losses. It is found that this $T_{\text{NL}}$ tax favors assets with less risk over those with higher risk and encourages portfolio diversification targeted at risk reduction, without regard to expected levels of portfolio return. Moreover, the burden of the tax is amplified for investment in put and call options, with the burden being greater when options are farther out of the money, and so incorporate more leverage and more risk. The high burden on options also translates into a burden on a synthetic division of risky asset ownership using put-call parity, with equity owners typically bearing substantially more risk than owners of direct interest in assets, and with synthetic owners bearing a greater aggregate tax burden than otherwise comparable owners of direct proportionate interests. This applies in particular to the case of debt financing, with separate debt and equity owners of an asset subject to a greater aggregate burden than they would be if they held the asset directly in proportionate shares.

The specific examples involving the $T_{\text{NL}}$ tax can be generalized in many respects and also applied to the current U.S. tax system. Taxes that are convex, like the $T_{\text{NL}}$ tax, generally burden risky investment relative to riskless investment, while taxes that are concave actually subsidize risk. Taxes that are subadditive, like the $T_{\text{NL}}$ tax, generally burden division of ownership relative to unified ownership, while taxes that are superadditive do the opposite. The current U.S. tax system does not fall into any one of these categories, but it has aspects of each, and to the extent it is creating unintended burdens or providing unintended incentives, it should be adjusted. In particular, the current subsidy for risky investment by low income taxpayers (created by concavity in the rate structure) seems to have little policy justification and should perhaps be revisited. In addition, the low cap on annual loss allowances creates

disincentives to division of ownership that could be diminished by increasing the total amount of losses allowed. The precise magnitude of changes that is appropriate in these and other cases depends on empirical questions about the number of taxpayers affected, as well as on political questions about what goals are actually desired. Nonetheless, the methodology can help inform the decision about changes by highlighting where burdens and subsidies are truly felt.

Beyond these several specific applications, the new conceptual framework laid out herein provides a valuable tool for enhanced understanding of the burden imposed by all types of taxes levied on risky returns. The methodology clarifies where burdens actually fall and can provide policymakers with guidance in determining what aspects of the tax system may need to be changed and to what degree. In ongoing related research, the author has begun applying the techniques developed herein to analyze various additional aspects of U.S. domestic taxation and also extending the techniques to the arena of international taxation and the U.S. taxation of foreign income. It is hoped that this work will deepen understanding of the true nature of tax burdens and enable policy decisions that are able to achieve their desired goals.

# A Appendix

## A.1 Convexity and Subadditivity of the $T_{\mathrm{NL}}$ Tax

Consider a tax that is proportionate on gains but does not permit any offset for losses. Thus, the tax due on an initial investment of $P_0$ that attains a final value of $P_1$ is

$$T_{\mathrm{NL}}(P_1 - P_0) = \tau \max(0, P_1 - P_0), \tag{A.1}$$

where $\tau$ is a specified constant rate of tax on gains.

**Theorem A.1** *The function $T_{\mathrm{NL}}$ is subadditive, so that*

$$T_{\mathrm{NL}}(x + y) \leq T_{\mathrm{NL}}(x) + T_{\mathrm{NL}}(y) \tag{A.2}$$

*for all $x$ and $y$, and it is also convex, so that*

$$T_{\mathrm{NL}}(\alpha x + (1 - \alpha)y) \leq \alpha T_{\mathrm{NL}}(x) + (1 - \alpha)T_{\mathrm{NL}}(y) \tag{A.3}$$

*for all $x$ and $y$ and all $0 \leq \alpha \leq 1$.*

*Proof:* To see that subadditivity holds, note that (A.2) is clearly true, and is actually an equality, if $x$ and $y$ are either both less than or equal to 0 or both greater than or equal to 0. The only other possibility is that one of $x$ and $y$ is less than 0 while the other is greater than 0. Without loss of generality, we may assume that $x$ is less than 0, and so $T_{\mathrm{NL}}(x) = 0$ and also $x + y < y$, which means that $T_{\mathrm{NL}}(x + y) < T_{\mathrm{NL}}(y)$. These combination of these results shows that $T_{\mathrm{NL}}(x + y) < T_{\mathrm{NL}}(x) + T_{\mathrm{NL}}(y)$ in this case, and (A.2) thus holds for all values of $x$ and $y$.

To see that convexity holds, note that (A.3) is equivalent to

$$T_{\mathrm{NL}}(\alpha x + (1 - \alpha)y) \leq T_{\mathrm{NL}}(\alpha x) + T_{\mathrm{NL}}((1 - \alpha)y),$$

and this inequality holds because of the subadditivity of $T_{\mathrm{NL}}$.

## A.2 Burden on Risky Assets Under a Convex Tax

Let $A$ and $B$ be risky and riskless assets, respectively, with prices $A_t$ and $B_t$ at time $t$. Assume that tax is levied in an amount equal to a function $T$ applied to the amount of gain experienced over an investment period lasting from time $t = 0$ to time $t = 1$. Thus, an investment of $A_0$ in the risky asset will result in a tax liability of $T(A_1 - A_0)$ at time $t = 1$, and similarly for an investment in the riskless asset.

**Theorem A.2** *If $T$ is a convex function, and if the risk-neutral distribution of $A_1$ has more than one point of nontrivial support, then the burden of the tax on an investment in the risky asset is greater than the burden on an investment of the same size in the riskless asset. The burden of the tax is measured as the certainty equivalent value of the tax, and so the inequality*

$$\mathrm{CEV}(P_0, A, T) > \mathrm{CEV}(P_0, B, T)$$

*holds, where $P_0$ is the amount of the initial investment.*

*Proof:* The certainty equivalent value for the $T$ tax for an initial investment of $P_0$ dollars in the risky asset is

$$\mathrm{CEV}(P_0, A, T_{\mathrm{NL}}) = \mathrm{PV}\left(\mathbf{E}\left[T\left(P_0\left(A_1/A_0 - 1\right)\right)\right]\right),$$

where the expectation is taken with respect to the risk-neutral distribution of $A_1$, the asset price at time $t = 1$, and where $\mathrm{PV}(x) = (B_0/B_1)x$ represents the present value of a payment of $x$ at time $t = 1$ calculated using the riskless rate of return.[41] An application of Jensen's inequality shows that

$$\mathrm{CEV}(P_0, A, T_{\mathrm{NL}}) \geq \mathrm{PV}\left(T\left(\mathbf{E}\left[P_0\left(A_1/A_0 - 1\right)\right]\right)\right),$$

and since $\mathbf{E}[A_1] = A_0(B_1/B_0)$, this is the same as

$$\mathrm{CEV}(P_0, A, T_{\mathrm{NL}}) \geq T\left(P_0\left(B_1/B_0 - 1\right)\right)(B_0/B_1).$$

This lower bound is the same as the certainty equivalent value for the $T_{\mathrm{NL}}$ tax for an initial investment of $P_0$ dollars in the riskless asset $B$, so that

$$\mathrm{CEV}(P_0, A, T_{\mathrm{NL}}) \geq \mathrm{CEV}(P_0, B, T_{\mathrm{NL}}),$$

with equality holding only when the distribution of possible payoffs for $A$ has support at only one value, meaning that $A$ is in fact riskless. Thus, the theorem follows.

## A.3 Burden on Debt Financing Under a Subadditive Tax

Let $A$ be a risky asset with ownership split into debt and equity components. The debt component, $D$, is a claim on the value of the asset up to a cap equal to a fixed 'principal" amount $K$. The equity component, $E$, is a claim on the excess of the value of the asset over

[41] The riskless rate of return is the same as the return on the riskless asset $B$.

$K$. Let $A_t$, $D_t$ and $E_t$ be the values of the asset and its components at time $t$, and note that the identity

$$A_t = D_t + E_t$$

holds for all values of $t$.

**Theorem A.3** *Consider two alternative situations: (i) one investor purchases $P_0(D_0/A_0)$ of the debt component of $A$ and another investor purchases $P_0(E_0/A_0)$ of the equity component of $A$; and (ii) a single investor purchases $P_0$ of the asset $A$ directly, for a total initial investment that is the same as the combined initial investments of the two investors in situation (i). In each situation, each investor makes his investment at time $t = 0$ and holds it until time $t = 1$, at which point tax is due. If $T$ is a subadditive tax, then the aggregate tax burden for the two investors is equal or greater in situation (i) to the burden in situation (ii). That is,*

$$\mathrm{CEV}(P_0(D_0/A_0), D, T) + \mathrm{CEV}(P_0(E_0/A_0), E, T) \geq \mathrm{CEV}(P_0, A, T), \tag{A.4}$$

*and so division of ownership into debt and equity between two individuals is burdened more heavily than direct ownership by a single individual. Moreover, if $T = T_{\mathrm{NL}}$, and if the support of the risk-neutral distribution of values for $A_1$ includes a neighborhood around $A_0$, then strict inequality holds in (A.4), and so the burden is strictly greater in situation (i).*

*Proof:* In situation (i), the aggregate tax burden for the two investors is

$$\begin{aligned} &\mathrm{CEV}(P_0(D_0/A_0), D, T_{\mathrm{NL}}) + \mathrm{CEV}(P_0(E_0/A_0), E, T_{\mathrm{NL}}) \\ &\quad = \mathrm{PV}\left(\mathbf{E}\left[P_0(D_0/A_0)(D_1/D_0 - 1)\right]\right) + \mathrm{PV}\left(\mathbf{E}\left[P_0(E_0/A_0)(E_1/E_0 - 1)\right]\right), \end{aligned} \tag{A.5}$$

where the expectation is taken with respect to the risk-neutral distribution of asset prices at time $t = 1$.[42] Because the expectation operator is linear, this is the same as

$$\mathrm{CEV}(P_0(D_0/A_0), D, T_{\mathrm{NL}}) + \mathrm{CEV}(P_0(E_0/A_0), E, T_{\mathrm{NL}}) \geq \mathrm{PV}\left(\mathbf{E}\left[P_0(A_1/A_0 - 1)\right]\right), \tag{A.6}$$

where the identity $A_1 = D_1 + E_1$ allows for simplification into an expression involving only $A$ values, rather than $D$ and $E$ values. This last expression is exactly the tax burden for the single investor in situation (ii), and so the main claim of the theorem follows.

To prove the remainder of the theorem, consider the case in which the tax function $T$ is $T_{\mathrm{NL}}$, and assume that the support for the risk-neutral distribution of values of $A_1$ includes a neighborhood around $A_0$. The inequality will be strict in this case if $D_1 - D_0$ and $E_1 - E_0$

[42] The values of $D_1$ and $E_1$ are determined by $A_1$, since $D_1 = \min(K, A_1)$ and $E_1 = \max(0, A_1 - K)$. Thus, the expectations depend only on the distribution of the values for $A_1$.

have opposite signs for some nontrivial range in the support of the distribution of values for $A_1$, because in this case the subadditive inequality that transforms (A.5) into (A.6) is a strict inequality. To demonstrate that the two quantities do indeed have opposite signs for a suitable range, write each as a function of $A_1$, so that $D_1 - D_0 = \min(A_1, K) - D_0$ and $E_1 - E_0 = \max(0, A_1 - K) - E_0$. Since $A_1$ has nontrivial support in the region $A_1 < K$, the value of $D_0$ is strictly less than $K$, and so $D_1 - D_0$ is an increasing function of $A_1$ that switches sign from negative to positive at a value of $A_1$ less than $K$. Also, $E_1 - E_0$ is an increasing function of $A_1$ that switches sign from negative to positive at a value of $A_1$ greater than $E_0 + K$, and in particular at a value greater than $K$. These observations together show that there is a neighborhood around $K$ in which $D_1 - D_0$ is positive and $E_0 - E_1$ is negative, and so strict inequality holds, as desired. Thus, if $A_1$ has nontrivial support in a neighborhood about $K$, we have

$$\mathrm{CEV}(D_0, D, T_{\mathrm{NL}}) + \mathrm{CEV}(E_0, E, T_{\mathrm{NL}}) > \mathrm{CEV}(A_0, A, T_{\mathrm{NL}}),$$

which means that a division of asset ownership with one investor holding a debt interest and the other holding an equity interest is subject to a greater tax burden than proportionate direct ownership of the asset.

## A.4 Burden on Investments in Options Under the $T_{\mathrm{NL}}$ Tax

Let $A$ be a risky asset, and let $C = C(K)$ be a European call option on $A$ with a strike price of $K$ and expiration at time $t = 1$. Thus, the final payoff of $C(K)$ is $\max(0, A_1 - K)$. Also, let $P = P(K)$ be a European put option on $A$ with a strike price of $K$ and expiration at time $t = 1$, so that the final payoff of $P(K)$ is $\max(0, K - A_1)$.[43]

**Theorem A.4** *Consider an investor who invests $P_0$ dollars at time $t = 0$ in European call options struck at $K$ and holds the options until time $t = 1$.*[44] *The burden under the $T_{\mathrm{NL}}$ tax is increasing in $K$, and it has a minimum value of $\mathrm{CEV}(P_0, A, T_{\mathrm{NL}})$ as $K \to 0^+$ and tends toward a maximum value of $\tau P_0$ as $K \to \infty$. Thus, the tax burden is always at least as great as that on an investment in a direct investment in the underlying asset, and it increases to a fraction $\tau$ of the present value of the initial investment as $K$ grows. If the investment is made in put options instead of call options, the reverse situation holds, with the tax burden decreasing in $K$, and it goes down from a maximum value of $\tau P_0$ as $K \to 0^+$ toward a minimum limit of $\tau \mathrm{CEV}(P_0, A, T_{\mathrm{NL}})$ as $K \to \infty$.*

[43] The notation for the put, $P = P(K)$ is similar to that frequently used in the text for an invested principal amount, $P_0$. The notation $P(K)$ will be used rather than $P$ to denote the put option with strike price $K$ whenever there is a possibility of ambiguity.

[44] Note that the amount of principal invested is fixed no matter what value of $K$ is chosen. Since $C_0(K)$, the price of a call option struck at $K$, is decreasing in $K$, this means that the investor buys more options if $K$ is larger, even though only the same total dollar investment is made.

*Proof:* The tax burden on an investment of $P_0$ dollars in $C(K)$ is

$$\begin{aligned}\text{CEV}(P_0, C(K), T_{\text{NL}}) &= \text{PV}\left(\mathbf{E}\left[T_{\text{NL}}\left(P_0(C_1(K)/C_0(K) - 1)\right)\right]\right) \\ &= \tau P_0(B_0/B_1)\mathbf{E}\left[\max(0, C_1(K) - C_0(K))\right]/C_0(K) \\ &= \tau P_0(B_0/B_1)\mathbf{E}\left[\max(0, (A_1 - K)/C_0(K) - 1)\right],\end{aligned} \tag{A.7}$$

where the expectation is taken with respect to the risk-neutral distribution of values for $A_1$. The first equality follows from the definition of CEV, the second equality follows from the definition of $T_{\text{NL}}$ as $T_{\text{NL}}(x) = \tau\max(0, x)$, and the third equality follows from the definition of $C_1(K)$ as $C_1(K) = \max(0, A_1 - K)$. The derivative of the tax burden with respect to $K$ can be written in terms of integrals and $\varphi(A_1)$, the risk-neutral distribution for values of $A_1$, as

$$\begin{aligned}\frac{\partial \text{CEV}(P_0, C(K), T_{\text{NL}})}{\partial K} = &-\frac{\tau P_0(B_0/B_1)}{C_0(K)}\int_{K+C_0(K)}^{\infty}\varphi(A_1)\,dA_1 \\ &+ \frac{\tau P_0(B_0/B_1)}{C_0^2(K)}\int_{K+C_0(K)}^{\infty}(A_1 - K)\varphi(A_1)\,dA_1\int_K^{\infty}\varphi(A_1)\,dA_1.\end{aligned}$$

This expression can then be rewritten as

$$\begin{aligned}\frac{\partial \text{CEV}(P_0, C(K), T_{\text{NL}})}{\partial K} = &\frac{\tau P_0(B_0/B_1)}{C_0(K)}\int_{K}^{K+C_0(K)}\varphi(A_1)\,dA_1 \\ &- \frac{\tau P_0(B_0/B_1)}{C_0^2(K)}\int_{K}^{K+C_0(K)}(A_1 - K)\varphi(A_1)\,dA_1\int_K^{\infty}\varphi(A_1)\,dA_1.\end{aligned}$$

The first integral in the subtracted second term in this final formulation has an integrand that is bounded above by $C_0(K)\varphi(A_1)$, and so the entire subtracted second term is less than the first term times a quantity that is less than 1.[45] As a result, the derivative of the tax burden with respect to $K$ is always non-negative, and so the tax burden is increasing with $K$, as claimed in the statement of the theorem.

To see how the tax burden tends in the limit, as $K \to \infty$, note that the final expression for $\text{CEV}(P_0, C(K), T_{\text{NL}})$ in (A.7) shows that

$$\lim_{K\to\infty}\text{CEV}(P_0, C(K), T_{\text{NL}}) = \lim_{K\to\infty}\frac{\tau P_0(B_0/B_1)\mathbf{E}[\max(0, A_1 - K - C_0(K))]}{C_0(K)}.$$

[45] This follows by replacing the $(A_1 - K)$ term in the integrand by $C_0(K)$ and cancelling this quantity with on factor of $C_0^2(K)$ in the denominator. The subtracted second term is then seen to be bounded above by the first term times $\int_K^{\infty}\varphi(A_1)\,dA_1$, which is less than 1.

Both the numerator and denominator of this last expression vanish as $K \to \infty$, and an application of L'Hopital's rule shows that

$$\lim_{K\to\infty} \mathrm{CEV}(P_0, C(K), T_{\mathrm{NL}}) = \tau P_0,$$

as claimed in the statement of the theorem. At the other extreme, as $K \to 0^+$, $C_0(K) \to A_0$, and so the final expression for $\mathrm{CEV}(P_0, C(K), T_{\mathrm{NL}})$ in (A.7) shows that

$$\lim_{K\to 0^+} \mathrm{CEV}(P_0, C(K), T_{\mathrm{NL}}) = \tau P_0 (B_0/B_1)\mathbf{E}[\max(0, A_1/A_0 - 1)] = \mathrm{CEV}(P_0, A, T_{\mathrm{NL}}).$$

The results for puts are proved in a similar fashion to those for calls. The formulation for the derivative of the tax burden with respect to $K$ is

$$\begin{aligned}\frac{\partial \mathrm{CEV}(P_0, P(K), T_{\mathrm{NL}})}{\partial K} = &-\frac{\tau}{P_0(K)} \int_{K-P_0(K)}^{K} \varphi(A_1)\, dA_1 \\ &+ \frac{\tau}{P_0^2(K)} \int_{K-P_0(K)}^{K} (K - A_1)\varphi(A_1)\, dA_1 \int_0^K \varphi(A_1)\, dA_1.\end{aligned}$$

The $(K - A_1)$ term in the first integrand of the added second term is dominated by $P_0(K)$, and making this replacement, one sees that the added second term is smaller than the absolute value of the first term. As a result, the derivative is always non-positive, as desired. The limiting values of the tax burden on a put investment strategy as $K \to 0^+$ and $K \to \infty$ are also derived in a fashion similar to the derivations for call strategies above.

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