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Classical, Modern and New Game Theory

by

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Abstract: This paper is a brief history of game theory with its main theme being the nature of the decision makers assumed in the various stages of its historical development. It demonstrates that changes in the "image of man" nourished the developments of what many believe to be progress in game theory. The first stage, *classical game theory*, is defined by John von Neumann's and Oskar Morgenstern's pioneering book "Game Theory and Economic Behavior" which introduced the concept of individual rational players and focuses on conflicting interests. The second stage, *modern game theory*, is defined by the *Nash player* who is not only rational but, at least implicitly, assumes that all players are rational to such a degree that players can coordinate their strategies so that a Nash equilibrium prevails. The third stage, *new game theory,* is defined by the *Harsanyi player* who is rational but knows very little about the other players, e.g., their payoff functions or the way they form beliefs about other players' payoff functions or beliefs. The Harsanyi player either plays a highly sophisticated epistemic game on the forming of beliefs or rests content with himself by imitating the observed successful behavior of other agents.

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1. Introduction

Is there progress in game theory? Do we know more today in this field than the scholars in the decade after John von Neumann and Oskar Morgenstern published their pioneering *Game Theory and Economic Behavior* in 1944. Or, did we only experience a change in style and language over the last fifty years?[1] The hypothesis of the following brief history of game theory is that the various stages of development are the result of different assumptions bout the nature of the decision makers underlying the alternative game theoretical approaches. The following text will not give a historical overview which aims for completeness.[2] Rather, it will trace the changes in the "image of man" implicit in the development of game theory and demonstrate some of consequences that follow.

We will distinguish three major stages in the development of game theory. The first one, *classical game theory*, is defined by John von Neumann's and Oskar Morgenstern's book. It introduced axioms for the concept of the individual rational player. Such a player makes consistent decisions in the face of certain and uncertain alternatives. But, such a player does not necessarily assume that other players also act rationally. In contrast, *modern game theory* is defined by the *Nash player* who is not only rational but assumes that all players are rational to such a degree that they can coordinate their strategies so that a Nash equilibrium prevails. The more recent, third stage in the development of game theory, *new game theory,* is defined by the *Harsanyi player.* This player is

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[1] In the categories of Paul Feyerabend, game theory is an art and thus follows its dynamic pattern: "There is no progress in art, but there are different styles and each style is perfect in itself and follows its own law. Art is the production of styles and art history is the history of their sequence" (Feyerabend, 1984, p.29).

[2] For example, we will not discuss whether John von Neumann or Emile Borel should get the credit for having introduced the minimax theorem: in any case, it was von Neumann who demonstrated its general validity (see Rives, 1975). Nor will we discuss the work of Ernst Zermelo, Denes König and Laszlo Kalmar on the finiteness and determinedness of the chess game. This work preceded the book of von Neumann and Morgenstern (1944), however, it focuses on analyzing of properties of chess and does not ask the question which characterizes modern game theory: how should a player behave to achieve a good result? (See Schwalbe and Walker, 2001.)

rational but knows very little about the other players, e.g., their payoff functions or the way they form beliefs about other players' payoff functions or beliefs. This limitation initiated two complementary strings of research: the more traditional one, based on a rational choice model, is characterized by the analysis of interactive *gedankenexperiments* about forming beliefs (i.e., epistemic games), while the second string follows an evolutionary approach where the agents rest content with themselves by imitating the observed successful behavior of other agents. The latter can be interpreted as the "rational conclusion" of the constrained cognitive capacity of the decision maker, on the one hand, and the complexity of the decision situation, on the other, or seen as the consequence suggested by the results of empirical research which challenge the rational choice model and its teleological background (see Frohn et al., 2001).

The discussion of these three stages forms the core of this paper. While much of the material is elementary and accessible to the non-specialist, the paper does contain some interpretive points that should be of interest to the advanced student of game theory.

2. Classical Game Theory and the Autonomously Rational Player

Game theorists consider the axiomatization of the utility function in the case of uncertainty a major contribution in von Neumann and Morgenstern (1944). It paved the ground for the modeling of rational decision-making when a decision-maker is faced by lotteries. Thereafter a utility function, $u_i(.)$, which satisfies the expected utility hypothesis, i.e.

(1) $u_i([A,p;B,1\text{-}p]) = pu_i(A) + (1\text{-}p)u_i(B)$

is called a von Neumann-Morgenstern utility function. In (1), A and B are events (or alternatives), p is the probability that event A occurs while 1-p is the probability of B occuring. Thus [A,p;B,1-p] is a lottery (or prospect). It is a notational convention to write [A,p;B,1-p] = A if p = 1 and [A,p;B,1-p] = B if p = 0. Of course, [A,p;X,1-p] = A for every alternative X if p = 1.

The probabilities p can be related to a model of relative frequencies and are, in this sense, objective and thus represent risk; or they can be subjective (i.e. expectations or beliefs) and thus represent uncertainty. The classical distinction between risk and uncertainty going back to Frank Knight (1921) appears, however, to be somewhat outdated today. For it does not seem to really matter in the end whether we believe in the objectivity of relative frequencies as an outcome of a random mechanism, or whether we derive our expectations from introspection and *gedankenexperiments*. One way or the other, they are all based on beliefs which reflect uncertainty and thus are subjective. If we follow this view and define rational behavior under uncertainty as maximizing expected utility in terms with (1), then our approach is *Bayesian*.

The utility values which the function $u_i(.)$ assigns to events (such as money, cars, or strawberries) are called payoffs. Because of (1) we do not have to distinguish between payoff and expected payoffs: if player i is indifferent between the lottery [A,p;B,1-p] and the sure event C then $u_i([A,p;B,1-p]) = u_i(C)$, i.e., the payoffs are identical. If $u_i(.)$ satisfies (1) then it is well defined as utility function of individual i up to a linear order-preserving transformation. That is, if $v_i(.) = a_i u_i(.) + b_i$ and $a_i > 0$ then $u_i(.)$ and $v_i(.)$ represent identical utility functions: thus $u_i(.)$ defines not a function but a family of functions and interpersonal comparison of utility is excluded because a_i and b_i are not determined.

The utility function of individual i can be linear, concave or convex in money - which coincides with risk neutrality, risk aversion, and risk affinity in so far as money defines the events of a lottery - or $u_i(.)$ can be related with money in a less rigid way without violating (1). There is, however, ample empirical evidence that individual behavior does normally not follow a pattern which is consistent with (1).[3] There are also strong intuitive arguments which challenge the adequacy of individual axioms which underlie the theory expressed in (1) such as the so-called *Allais paradox*. Later Nobel Laureate Maurice Allais (1953, p 527) demonstrated the proposed inconsistency of the axioms of the von Neumann Morgenstern utility theory by means of the following example:

(1) People are asked whether they prefer alternative A or alternative B where

Alternative A: 100 millions for sure

$$\text{Alternative B: } \begin{cases} \text{a chance of 0.1 to win 500 millions} \\ \text{a chance of 0.89 to win 100 millions} \\ \text{a chance of 0.01 to win nothing} \end{cases}$$

(2) People are asked whether they prefer alternative C or alternative D where

$$\text{Alternative C: } \begin{cases} \text{a chance of 0.11 to win 100 millions} \\ \text{a chance of 0.89 to win nothing} \end{cases}$$

$$\text{Alternative D: } \begin{cases} \text{a chance of 0.1 to win 500 millions} \\ \text{a chance of 0.9 to win nothing} \end{cases}$$

The money values are probably in "old" French francs. The expected values of A, B, C, and D are (measured in millions) 100, 139, 11 and 50, respectively.

Allais argues that for a large number of people, especially for those who are averse against taking risk, one observes that they prefer A to B and D to C. However, von Neumann Morgenstern utility theory suggests that if A is preferred B then C is preferred to D. In order to see this, we write these

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[3] See the seminal paper of Kahneman and Tversky (1979) for a prominent critique of the von Neumann Morgenstern utility function and Machina (1987) for a summary review of the discussion.

preference relations in terms of the von Neumann Morgenstern utility function of an agent i:

"A preferred to B" implies: $u_i(100) > 0.1u_i(500) + 0.89u_i(100) + 0.01u_i(0)$

"C preferred to D" implies: $0.11u_i(100) + 0.89\,u_i(0) > 0.1u_i(500) + 0.9u_i(0)$

Both inequalities can be reduced to $0.11u_i(100) > 0.1u_i(500) + 0.01u_i(0)$. Thus "A preferred to B" implies "C preferred to D". Consequently, "D preferred to C" is inconsistent with "A preferred to B" and corresponding behavior violates the expected utility hypothesis (1).

There are, however, also strong arguments in favor of (1) and the underlying axioms formalized in von Neumann and Morgenstern (1944). Firstly, there is empirical evidence that people tend to correct their behavior if they are aware that it deviates from (1) or one of its implications. Secondly, the generalization of alternative approaches to decision-making under uncertainty (such as the *prospect theory* of Kahneman and Tversky (1979) and the *similarity approach* of Rubinstein (1988)) are also criticized on the basis of contradicting empirical results and implausibility of underlying assumptions. Moreover, the alternative approaches tend to be more complicated than the theory behind (1) and therefore more difficult to apply to real life decision-making and textbook analysis. This is perhaps the main reason why game theorists stick to the von Neumann-Morgenstern utility function when it comes to decision-making under uncertainty. The maximization of such a utility function defines the *rational player* in game situations, i.e. if the outcome of a choice depends on the action of at least two agents and the agents, in principle, put themselves into the shoes of the other agents when they make their decisions because they know of the interdependence of decisions.

There are however many ways to specify this knowledge and thus the image which a player has of the other player(s). Von Neumann and Morgenstern (1944) assumed that a player i does not expect that player j is necessarily rational: j's behavior may violate the theory embedded in (1) and its implications. In their theory of games, they propose that players should act rational even under the assumption that other players are irrational, i.e. inconsistent with (1): "... the rules of rational behavior must provide definitely for the possibility of irrational conduct on the part of others. ... In whatever way we formulate the guiding principles and the objective justification of 'rational behavior,' provisios will have to be made for every possible conduct of 'the others'" (p. 32). To characterize this proposition we will speak of *autonomously rational players* in the theory of von Neumann and Morgenstern.

2.1 The Minimax Theorem

It may come somewhat of a surprise, but von Neumann and Morgenstern's theory provides convincing results only if we have a situation in which there is *pure conflict of interest* between two players and the decision situation can be modeled as a zero-sum game. For example, if we assume that the payoff (bi-)matrix in Figure 1 is specified by the payoff values $a = -\alpha$, $b = -\beta$, $c = -\gamma$, and $d = -\delta$, then it describes a zero-sum (two-by-two) game where player 1 has the pure strategies s_{11} and s_{12} and player 2 has the pure strategies s_{21} and s_{22}.

Figure 1: *Generalized two-by-two game*

	s_{21}	s_{22}

s_{11}	(a,α)	(b,β)
s_{12}	(c,γ)	(d,δ)

In principle, the definition of utility functions given in (1) does not allow for interpersonal comparison of utility as implied by the zero-sum property. However, if there is pure conflict of interest between two players then the assumption that a utility gain to player 1 is a utility loss to player 2, and vice versa, seems appropriate. Note that, if the payoff values of the two players in each cell add to the same constant value, then the game is equivalent to a zero-sum game and can, without loss of information, be transformed into such a game.

Given a zero-sum game, von Neumann and Morgenstern (1944) suggest that each player will choose his maximin strategy. Thus player 1 looks for the minimum payoff in each line and then earmarks the strategy which is related to the highest payoff of these (two) minima while player 2 does likewise for his payoffs in each column. If the earmarked value of player 1 and the earmarked value of player 2 add up to zero, then the corresponding strategy pair characterizes the solution and the related payoff pair describes the outcome.

If the earmarked values do not add up to zero, then player i (i = 1, 2) will randomize on his strategies such that the expected value is independent of whether the other player chooses his first or second strategy or any mixture of these strategies. For instance, if player 1 chooses his first strategy with probability p and player 2 chooses his first strategy with probability q, then p and q are determined by the two equalities:

$$pa + (1-p)c = pb + (1-p)d \text{ and } q\alpha + (1-q)\beta = q\gamma + (1-q)\delta$$

Solving these equalities, we get

(2) $$p° = \frac{d - c}{a - b - c + d} \quad \text{and} \quad q° = \frac{\delta - \beta}{\alpha - \beta - \gamma + \delta}$$

It is easy to show that the (expected) payoff player 1 is equal to the negative of the payoff of player 2 if they choose their corresponding first strategies with probabilities p° and q°. This is the essence of the so-called *minimax theorem* of von Neumann and Morgenstern which says that, given a two-person zero-sum game, there is *always* a pair of strategies, either in pure or mixed strategies, such that the maximin payoff equals the minimax payoff of player 1. Note that in two-person zero-sum games the maximin payoff of player 2 with respect to his own payoff values is identical to the minimax value with respect to the payoffs of player 1. (Because the payoffs of player 2 are the negative values of the payoffs of player 1, it is sufficient to specify the payoffs of player 1 only.)

2.2 Limitations of Classical Game Theory

Baumol (1972, p. 575) summarizes the classical view on game theory which derives from the minimax theorem: "In game theory, at least in the zero-sum, two-person case, there is a major element of predictability in the behavior of the second player. He is out to do everything he can to oppose the first player. If he knows any way to reduce the first player's payoff, he can be counted upon to employ it." However, the *minimax theorem* loses its power if the players' interests do not contain pure conflict and the zero-sum modeling becomes inappropriate. This is particularly the case if strategic coordination problems become eminent. For instance, let's assume that $a > 0$, $\alpha > 0$, $d > 0$, $\delta > 0$, and all other payoffs in Figure 1 are zero. Then the matrix in Figure 1 represents a *variable-sum game* and the minimax theorem does not, in general, apply anymore. Assume further, player 1 and 2 have to choose their strategies simultaneously - or in such a way such that they cannot see what the other player has chosen. Then a player has to solve the problem of how to coordinate his strategy with the strategy of the other player in order to gain positive payoffs. The fact that the theory of von Neumann and Morgenstern says little about coordination problems, in particular, and variable-sum games, in general, concurs with the problem that the guiding hand of self-interest becomes weak in strategic situations if there is no pure conflict of interest and players have difficulties to form expectations about the behavior of their fellow players.

It is not surprising that the textbook representation of game theory of the 1950s and still in the early 1960s focused on the two-person zero-sum game and problems of how to calculate the maximin solution if players have more than two pure strategies (see, e.g., Allen, 1960). An exception is the ingenious book by Luce and Raiffa (1957) which is still a great source of inspiration for game theorists.

The assumption of a pure conflict of interest seems also questionable if there are more than two players. If we try to formulate a zero-sum game for three players then the problem becomes rather obvious. Moreover, in the case of more than two players there is a potential for coalitions. Von Neumann and Morgenstern (1944) developed the concept of the characteristic function in order to express the value of a coalition. They also suggested a solution concept for the case of more than two players which may take care of coalition formation: they simply called this concept *solution*.[4] However, neither does its application give an answer which coalition will form nor does it determine the payoffs which the individual players get in the course of the game.

[4] It is identical with the concept of stable sets of modern game theory (see Owen, 1995, pp. 243-249).

It is fair to mention that even more than fifty years later the existing theories of coalition formation provide answers to these two problems only if the coalition games are rather specific and the competing theories generally provide divergent answers. However, for the case of variable-sum games and games with more than two players (if they do not form coalitions) a very promising solution concept has been suggested: the Nash equilibrium and its refinements.

3. Nash Equilibrium and Modern Game Theory

In his doctoral dissertation, John Nash (1951) proved that for finite game - a game with a finite number of pure strategies assigned to each member of a finite set N of n players - an equilibrium s* exists. s* is a strategy vector $s^* = (s_1^*, ..., s_i^*, ..., s_n^*)$, where s_1^* is the strategy of player, s_i^* is the strategy i, and s_n^* is the strategy of player n, such that $u_i(s_1^*, ..., s_i^*, ..., s_n^*)$ ### $u_i(s_1^*, ..., s_i, ..., s_n^*)$ for all (pure or mixed) strategies s_i which player i can choose and for all players i in N. Thus, if player i chooses s_i^* then he cannot achieve a higher payoff by choosing an alternative strategy s_i, *given that the other players choose their strategies in accordance with vector* s*. We say that the strategies in s* are *mutually best replies* to each other and, consequently, s* is a *Nash equilibrium*.

If mutually best replies are the result of decisionmaking then, obviously, the Nash equilibrium assumes that players are rational *and* that they expect the other players be also rational. This assumption is quite different from the autonomously rational player which characterizes the theory of von Neumann and Morgenstern. In fact, the underlying assumption is more general and even stronger: "...an equilibrium strategy describes a player's plan of action as well as those considerations which support the optimality of this plan" (Rubinstein, 2000, p.77). One of the considerations is that the other players are rational - another is that the other players choose the (equilibrium) strategies such that the chosen strategy is a best reply.

3.1 Dominant Strategies and Refinements

It is straightforward that Nash equilibrium and maximin solution proposed by von Neumann and Morgenstern are identical for zero-sum games. If we assume c > a > d > b and $\beta > \alpha > \delta > \gamma$ in Figure 1, then the matrix represents a variable-sum game and the strategy pair (s_{12},s_{22}) is a Nash equilibrium of this game. In fact, because of the assumed preference relation the matrix describes a specification of the famous prisoners' dilemma game as both players have a dominant strategy and the Nash equilibrium is inefficient inasmuch as it is payoff dominated by the result of playing strategies s_{11} and s_{21} - however, these strategies are strictly dominated by the equilibrium strategies and rational players will not choose them if the game is played only once and independent of any other decision situation. It is therefore sufficient to rely on dominant strategies to obtain the strategy choices (s_{12},s_{22}) for rational decisionmakers.

It is obvious that the Nash equilibrium is identical with the equilibrium in strictly dominating strategies if the latter exists. However, let us look at the matrix in Figure 2 which is characterized by the weakly dominating strategies s_{12} and s_{22}. The game has two Nash equilibria: the strategy pairs (s_{11},s_{21}) and (s_{12},s_{22}). The latter, however, is not very convincing because it contains weakly dominated strategies. A player cannot do worse by choosing his first instead of his second strategy.

Figure 2: *Nash equilibrium in weakly dominated strategies*

	s_{21}	s_{22}
s_{11}	(1,1)	(0,0)

s_{12}	(0,0)	(0,0)

We can get rid of the Nash equilibrium (s_{12},s_{22}) if we apply Reinhard Selten's concept of trembling-hand perfectness.[5] In order to test whether the equilibrium (s_{12},s_{22}) is trembling-hand perfect, we assume that player 1 considers the possibility that with a small probability ε the hand of player 2 may tremble and 2 selects strategy s_{21} instead of s_{22}. Then player 1's expected value of strategy s_{11} is higher than of s_{12} and he will be very hesitant to choose the latter although it is a Nash equilibrium strategy. In the end, both players will choose their first strategies with probabilities 1 and *do the right thing for the wrong reason.*

3. 2 The Multiplicity of Nash Equilibria

It seems that Selten's trembling-hand perfectness is a rather powerful concept although a trembling-hand is rather peculiar in a game of complete information when players are rational (i.e. both players know the game matrix as it is printed in Figure 2). Trembling-hand perfectness is not, however, very helpful for discriminating among the three Nash equilibria in the game of Figure 3: the strategies pairs (s_{11},s_{22}) and (s_{12},s_{21}) and the pair of mixed strategies given by $p^* = q^* = 1/2$. The equality $p^* = q^*$ is due to the symmetry of the game. If players choose these mixed strategies none of them can reach a higher payoff by choosing a different strategy, i.e., p* and q* are mutually best replies. The corresponding payoffs of the equilibrium (p*, q*) are 1.5 for each player - therefore the mixed strategy equilibrium is inefficient because both players are better off by choosing their first strategies.

[5] Reinhard Selten (1975) introduced this concept as a refinement of the Nash equilibrium.

Figure 3: *Chicken game*

	s_{21}	s_{22}
s_{11}	(2,2)	(1,3)
s_{12}	(3,1)	(0,0)

How we should discriminate among the three Nash equilibria of this game and how players can coordinate their choices so as to achieve one of them, is not obvious at all. For example, both equilibria in pure strategies are trembling-hand perfect. Of course, if player 1's hand trembles and he mixes his pure strategy by another probability than p* then player 2 has a best reply which is a pure strategy. By this standard, the mixed strategy Nash equilibrium does not appear to be a likely candidate for describing the result of the game. Moreover, this equilibrium is payoff-dominated by the strategy pair (s_{11},s_{21}) which gives a value of 2 to each player. However, (s_{11},s_{21}) is not a Nash equilibrium.

The nicety of the mixed strategy equilibrium is its symmetry; in contrast, the two pure strategy equilibria of the game in Figure 3 discriminate against one of the players. The symmetry seems advantageous if players have to coordinate their strategies without knowing the other player's strategy.

Obviously, it is not sufficient to assume that players are rational and maximize their utility as defined by a function which satisfies (1) to guarantee that a Nash equilibrium actually occurs. If there are more than one Nash equilibrium then we have to make rather strong assumptions so that the players can coordinate their strategies on a Nash equilibrium.[6] It is not sufficient to assume that player 1 assumes that player 2 is rational, and vice versa. Under this

[6] Tan and da Costa Werlang (1988) have analyzed these assumptions on an axiomatic basis.

assumption, the strategy pair (s_{11},s_{21}) and a payoff of 0 to each of the two players - as outcomes of coordination failure - cannot be excluded. With a shot of irony Mailath (1998, p.1351) concluded that "the consistency in Nash equilibrium seems to require that players know what the other players are doing." The discussion of the epistemic conditions for Nash equilibria below will demonstrate that this statement does not fall far from the truth.

It is not easy to see how players should manage to coordinate on one of the efficient pure strategy equilibria of the chicken game in Figure 3, even if they can communicate before they choose their strategies, given that this game is played only once. Aumann (1985) contains a nice example of 50-person game with non-cooperative coalition formation which demonstrates that pre-play communication does not help to select a favourable equilibrium outcome: "Three million dollars are to be divided. Each of the players 1 through 49 can form a two-person coalition with player 50, which *must* split 59:1 (in favor of 50), yielding the 'small' partner $50,000. The only other coalition that can get anything consists of all the players 1 through 49, which *must* split evenly, yielding each player about $61,000." What will happen if player 50 calls player 1 and asks him to join with him and get a share of $50,000? Will player 1 reject this offer trying to round up the fellow players 2 to 49 in order to form an alternative coalition, S, and get the higher share of $61,000? The latter only makes sense if player 1 can be sure that none of his fellow players will be phoned by player 50 to get lured into a coalition with him.

Player 1 must have very strong opinions about his fellow players and this must hold for each individual player in the group of players 1 to 49 so that player 1's beliefs are justified. If one of them does not trust the integrity of every single player to reject the offer of player 50 then coalition S is not feasible. In this game situation, we can expect that there will be more players who do *not* trust the others than those who do - in fact, it might be hard to find a single player who will resist a proposition made by player 50. If players can converse with each other and make tentative, however, non-binding agreements before decisions have to be made, it could well be that more than one gets "fooled" into the belief that coalition S will form, but not *every* potential member of S will be a "fool" and believe in the forming of this coalition. Of course, whether a player is a fool depends what other players believe and do.

3.3 Unprofitability of Mixed Strategy Equilibria

Not only is the multiplicity of Nash equilibria a drawback of this concept if we want to derive principles of individual choice from it - or perhaps even try to make use of it for forecasting or proposing rational decisions. If we assume that the payoffs in Figure 1 (above) satisfy one of the following two orderings

A.1 $a > c$, $a > b$, $d > b$, $d > c$
$\beta > \alpha$, $\beta > \delta$, $\gamma > \alpha$, $\gamma > \delta$

A.2 $b > a$, $b > d$, $c > a$, $c > d$
$\alpha > \beta$, $\alpha > \gamma$, $\delta > \beta$, $\delta > \gamma$

then both the Nash equilibrium and the maximin solution are in mixed strategies. The maximin solution is defined by p° and q° as in (2) above while the Nash equilibrium strategies are

(3) $$p^* = \frac{\delta - \gamma}{\alpha - \beta - \gamma + \delta}$$

$$q^* = \frac{d - b}{a - b - c + d}$$

Note that the Nash equilibrium strategy of player 1, p*, is exclusively determined by the payoffs of player 2 and the Nash equilibrium strategy of player 2, q*, is exclusively determined by the payoffs of player 2. Consequently, if there are changes in the payoffs of player 1 but A.1 or A.2 still apply,[7] then the equilibrium behavior of player 2 is affected while p* remains unaffected. This rather paradoxical result has produced a series of applications, however, some are of rather dubious empirical value. (See Frey and Holler, 1998, and a review of related results therein.) Moreover, it is easy to show that the payoffs in the Nash equilibrium (p*, q*) and the maximin solution (p°, q°) are identical (Holler, 1990). It is therefore not "profitable" to play the Nash equilibrium strategy, and thus rely on the rather specific rationality of the other player required by the Nash equilibrium, because a player can assure himself the identical value by choosing maximin, *irrespective* of the strategy choice of the other player. In terms of Harsanyi (1977, p. 104-107) we can state the "unprofitability of Nash equilibrium" in two -by-two games if both Nash equilibrium and maximin solution are in mixed strategies.

In case of unprofitable mixed-strategy equilibria, Harsanyi (1977, p. 125) strongly suggests that players choose maximin strategies instead of trying to reach an equilibrium. As Aumann (1985, p. 668) concludes from studying an unprofitable mixed strategy equilibrium: "Under these circumstances, it is hard to see why the players would use their equilibrium strategies."

Perhaps an answer to this question could be found in an evolutionary context: Andreozzi (2001) shows for a dynamic replicator model (see section 5 below) of the game in Figure 1 for which payoffs are constrained by assumption A.2 that the *averages* of strategy choices concur with p* and q*. He also shows that maximin strategies and Nash equilibrium strategies can co-exist in a dynamic model.

3.4 Incomplete Information and Mixed Strategy Equilibrium

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[7] There are weaker conditions such that the Nash equilibrium is mixed and the maximin solution might be in pure strategies.

Another answer to the Aumann's question "why the players would use their equilibrium strategies" can be found in Harsanyi (1973). Harsanyi proposes a model under which both players' actual behavior seems to coincide with the strategies of the mixed-strategy equilibrium.[8] The model assumes that players derive payoffs which are subject to small random fluctuations: each player knows his true payoff values, but player i is only informed of the mean of player j's payoffs. As a consequence, both players will in fact use only pure strategies in the equilibrium.

The reasoning behind this result is as follows: Suppose that players 1 and 2 know the payoffs in Figure 1 only *approximately* and ordering A.2 applies. This defines a game G*. Given the random fluctuation in payoffs, players will be close to being indifferent between their two pure strategies in G* if they assume that the other player will play his first strategy with a probability p* or q*, respectively. At some times their first strategies give them more utility, while at other times the second strategy is more profitable. In the perturbed G* the equilibrium strategies of both players are: "Always use your pure strategy yielding a higher - even if only slightly higher - payoff; but if both of your pure strategies yield the same payoff then you may use either of them" (Harsanyi, 1992, p.58). Note that these strategies do not contain any randomization. Moreover, if the probability distributions of the random fluctuations are continuous then the probability that the two pure strategies of a player yield the same payoff is zero and the possible indifference of the players can be ignored.

Given the pure-strategy equilibrium of the G*, the random fluctuations in the payoffs will make, on the one hand, strategy s_{11} with a probability slightly greater than p* more profitable to player 1 than his second strategy, s_{12}.[9] On the

[8] See also Harsanyi (1992) for an introductory discussion.

[9] Recall that (p*, q*) represents the mixed strategy equilibrium of the game in Figure 1 in accordance with (3), here specified by payoffs corresponding to A.2.

other hand, s_{12} will be more profitable with a probability slightly smaller than (1- p*). Consequently, player 1 will choose s_{11} with a probability slightly larger than p* and s_{12} with probability slightly smaller than (1- p*). Likewise player 2 will choose s_{21} with a probability slightly smaller than q* and s_{22} with probability slightly larger than (1- q*).

If the amplitudes of the random fluctuations are small, the probabilities p* and q* are good estimates for the expected equilibrium behavior: if the amplitudes of the fluctuations go to zero, the probabilities for choosing s_{11} and s_{21} will converge to p* and q*, respectively. However, in the equilibrium of G*, both players will play pure strategies and the equilibrium will be *strict*: deviations would inflict utility losses depending on the degree of fluctuation. If the game is repeated and payoffs obteined are randomly distributed in each round, *outside observers* can interpret the relative frequencies for the alternative pure strategies as players playing mixed strategies.

The random fluctuations on the payoffs represent the *incomplete information* of the players with respect to the other player's payoffs, i.e. his or her identity as a player. If players do not know their own strategy sets or the strategy sets of the other players, then this also implies incomplete information: again the identity of at least one player is not determined. This also means that from the perspective of at least one player the rules of the game are not well defined since the rules of a game determine the strategy sets available for the various player. However, incomplete information about the rules of the game can be seen as higher order of incomplete information and "different players may have different beliefs about the game that is played" (Vilks, 2001, p. 89). If, however, the identity of the players is undetermined, the game situation is also undetermined and cannot be analyzed in terms of game theory proper. This was the state of the art before Harsanyi (1967/68) demonstrated how to describe a game of *incomplete information* in terms of a game of *imperfect information* and then find a solution to the latter.

A game is characterized by perfect information if a player knows all the decisions made by him and other players before he has to make a choice. (This is the case if all information sets have only one element.) Cases of imperfect information prevail if a player forgets how he has decided earlier in the game or if he cannot see the choices of the other players which are simultaneous, at least in *strategic time*. Now, Harsanyi's idea was that, if player 1 does not know the payoffs of player 2, let us assume that "nature" decides the payoffs of player 2, i.e. what *type of player* he is. However, we further assume that player 1 cannot see the move of nature which makes it an imperfect information game.

If player 1 is a Bayesian decision maker and has some prior information about the possible types of player 2, he will assign probabilities $p_1,...,p_m$ if there are, in his view, m possible types of player 2. If he has no specific information about the probabilities of the various types of player 2, but still distinguishes m possible types, player 1 may well apply the *principle of insufficient reason* and assign p = 1/m to each of the m types of player 2. If the probabilities are given - or, in a continuous model, the density function over the type interval - then player 1's equilibrium decision coincides with maximizing the expected utility under the assumption that each type of player 2 chooses an equilibrium strategy.

Correspondingly, a mixed strategy of player i can be interpreted as a *conjecture* which expresses the player j's uncertainty about the player i's strategy choices while player i will, in fact, choose a pure strategy. Thus, the pair of equilibrium strategies (p*, q*) above represents a pair of *consistent conjectures* such that any pair of pure strategies concurs with them - while the selected pure strategies are in general not best replies to each other.

4. The Forming of Beliefs

Not surprisingly, mixed strategy equilibria have been interpreted as the *belief* held by all other players concerning a player's action (e.g., Aumann, 1987). The uncertainties implied by the mixed-strategy equilibrium have been viewed as an expression of the lack of certainty which characterizes the underlying strategic situation rather than the result of intentional plans and strategy choices of rational players. Given the beliefs incorporated in a mixed strategy equilibrium, all actions with strictly positive probabilities are optimal and a player is "free too choose" any pure or mixed strategy - and regret after the decisions of the other players have been disclosed that he has chosen the "wrong" strategy.

Strategic uncertainty, as in mixed strategy equilibrium or in cases of a multiplicity of equilibria, are however but one instance where players, explicitly or implicitly, form beliefs. A second instance, as we have seen, is given if the information of a player is *incomplete*. Strategy-related or payoff-related uncertainties are of *first-order* and players have to form expectations - i.e., beliefs - about the facts which create these uncertainties, in order to make "adequate" decisions. Whether a decision is adequate, however, will depend on the decision and thus on the beliefs of the other players and the beliefs which they hold on the beliefs of the other players. The uncertainty about beliefs can be modeled in a similar fashion to uncertainty about payoffs, etc. by introducing additional types of players. If players are uncertain about what beliefs other players hold about other players, the game is characterized by *second-order uncertainty* - which is, in this case, the result of first-order uncertainty.

4. 1 Beliefs and Solution Concepts

As we have seen, *second-order uncertainty* can also be the result of strategic uncertainty which is the result of the players' mutually interdependent reciprocal expectations about each other's behavior.[10] Solution concepts such as Nash equilibrium, trembling-hand perfect equilibrium or maximin are tools to *reduce* this interdependency and to get around the problem of discussing beliefs. In cases of zero-sum games, the prisoners' dilemma or the game in Figure 2 these tools work quite successfully. However, the trembling-hand perfect Nash equilibrium, applied to eliminate the Nash equilibrium in weakly dominated strategies in Figure 2, makes rather strong assumptions on the beliefs of the players. Moreover, these beliefs are inconsistent with the outcome which proposes that the weakly dominant strategies are played with certainty. It is the assumed "trembling of the hand" which creates the uncertainty in a game of complete information which allows the application of the Bayesian principle (i.e., the maximization of expected utility) which sorts out weakly dominated strategies to be considered as outcomes of the game, even if they are rationalizable as Nash equilibrium strategies.

In the standard case, Bayesian decisionmaking is applied more directly to games of incomplete information, when the beliefs related to the incompleteness are transformed by the move of "nature" into probabilities à la Harsanyi (1967/68) so that the expected utility calculation (1) applies. This calculation, combined with a game structure, defines a Bayesian equilibrium which consists of mutually best replies of strategies for all type of players,[11] qualified by the probability estimates that a specific type prevails. Bayesian decision makers are assumed to revise their beliefs by conditioning on new information. However, there are hardly any constraints on these probability estimates. A possible restriction is given by the assumption of *common priors* which assumes that the probability estimates derive from empirical observations or abstract reasoning which are common for all players.

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[10]The Nash equilibrium is "based on the idea that a rational theory should not be a self-destroying prophecy which creates an incentive (not) to deviate for those who believe in it" (Selten, 1987, p.79).
[11]Note that a type of player can be "justified" by incomplete information on strategy sets, payoffs or beliefs.

Alternative restrictions define the sequential equilibrium as introduced by Kreps and Wilson (1982a). In the case of a two-person game, it is a combination of a pair of strategies which are mutually best replies and a pair of consistent beliefs that reflect each player's probability *assessment* over the nodes in each information set. This reflects what players believe what has happened in the course of the game so far. Beliefs are consistent if they are consistent with the equilibrium strategies and satisfy Bayes' rule whenever it applies. Consequently, "starting with *every* information set, the player whose turn it is to move is using a strategy that is optimal for the remainder of the game against the hypothesized future moves of its opponent (given by the strategies) and the assessment of past moves by other players and by 'nature' " (Kreps and Wilson, 1982b, p. 257). This is also required for information sets that will not be reached if players follow the prescription of the equilibrium strategies.

On the one hand, sequential equilibria are subgame perfect,[12] while, on the other, the set of trembling-hand perfect equilibria form a subset of sequential equilibria. The difference is in the assumption about the information and beliefs of the players. Obviously, the beliefs in the trembling-hand perfect equilibria are more restrictive than in sequential equilibria.

4. 2 Epistemic Analysis

The refinements of the Nash equilibrium are characterized by more or less informal assumptions about how players form and revise their beliefs. *Interactive*

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[12] Given a game Γ in extensive form (described by a game tree). A Nash equilibrium pair (σ, μ) is subgame perfect if, wherever Γ decomposes (at a node X), the corresponding restrictions of σ and μ are an equilibrium pair for the "remaining" subgame Γ_X (Owen, 1995, p.169). In games of perfect information, subgame perfectness implies that in each decision node the corresponding player chooses a partial strategy (which is part of his subgame perfect strategy) such that it is a best reply to the decisions of the other players and his own decisions that follow. If the game is *finite*, subgame perfect equilibria are determined by *backward induction.* (See, e.g., the centipede game in Figure 5 below.)

epistemology[13] is the formal analysis which helps to clarify the assumptions about the beliefs and reasoning of players that are implicit in the various solution concepts, i.e., what players know and believe about the facts of the game (players, strategies and payoffs) and each other's knowledge and beliefs. Interactive epistemology is about rigorously and explicitly formulating assumptions about the knowledge, the beliefs, and the rationality of the players and thus directly tackles the core issue of game theory: the formation of expectations when decisions are interdependent. On the one hand, it generalizes the idea of a solution concept and, on the other hand, it substitutes, when it seems appropriate, the choice of a specific solution concept, and thus can possibly add new insight about the strategic contents of game situations - particularly for games of incomplete information and for repeated games in which the history may serve to signal the intentions of the players in future rounds.[14] From this perspective, solution concepts such as Nash equilibrium and its refinement are just shorthand concepts of the strategic interaction and the related belief formation of a game situation.

Applying interactive epistemology, a state of a game comprises a profile of strategies, which encode each player's potential to act, and epistemic types which summarize each player's dispositions to hold conditional beliefs about the other players' strategies and types. The latter are expressed by means of conditional probabilities which satisfy Bayes' rule whenever it applies. Thus the players dispositions to hold beliefs is described by a *conditional probability system.* Higher-order beliefs, i.e., beliefs about beliefs, beliefs about be beliefs about beliefs, and so on, are a constituent element and reflect corresponding levels of uncertainty. An *epistemic type* is, in analogy to the modeling of incomplete information on strategy sets or payoffs, a complete and explicit

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[13]See Aumann (1999a,b), Battigalli and Siniscalchi (1999), and Battigalli and Bonanno (1999) for this concept.

[14]To illustrate the latter: what does it mean if the first player does not choose "down" at the initial "history" of the centipede game? For the centipede game, see Figure 5 below.

description of an agent's conditional beliefs of arbitrary order. Thus we can model the incomplete information on belief systems by sets of *epistemic types* - or, equivalently, by hierarchies of conditional beliefs.

In addition to the constraints through applying Bayes' rule, a conditional probability systems is assumed to satisfy the notion of *common certainty of opponent's rationality.* Although the intuition behind this concept is self-explanatory, however, its formal elaboration is beyond the scope of this paper. Instead I will illustrate major features of the concepts of epistemic analysis by means of an example borrowed from Battigalli and Siniscalchi (1999). Figure 4 illustrates a signalling game. The set of payoff-relevant types of players 1 and 2 are given by $\Theta_1 = \{\theta_{11},\theta_{12}\}$ and $\Theta_2 = \{\theta_2\}$, respectively. Since Θ_2 is a singleton there is no payoff-related uncertainty about player 2. However, it is player 1's private information whether he is of type θ_{11} or θ_{12} . This models the incomplete information of player 2 with respect to the payoffs of player 1 by the imperfect information of whether "nature" has chosen θ_{11} or θ_{12}. The set of pairs of types is $\Theta_1 \times \Theta_2 = \{(\theta_{11},\theta_2), (\theta_{12},\theta_2)\} = \{\theta',\theta''\}$ where × is the Cartesian product. Player 2's belief about the opponent's payoff-relevant type will be specified within an epistemic model.

Figure 4: Signalling Game

The set of (final) outcomes is $\{(L), (R,u), (R,d)\} \times \{\theta',\theta''\}$. $H = \{\phi, (R)\}$ is the set of partial histories where ϕ represents the *empty history* (nothing has been observed) and (R) expresses the intermediate outcome "player 2 observed that player 1 had chosen R" For each type θ_i and each partial history *h* (which is an element of H), the Cartesian product $\{h\} \times \{\theta_i\}\times \Theta_i$ corresponds to an information set for player i in Figure 4. For example, $I = \{((R), \theta_2, \theta_{11}), ((R), \theta_2, \theta_{12})\}$ describes the information set of payer 2.

In principle, we are interested in players' conditional beliefs for each commonly observed partial history *h*. In the game of Figure 4, however, only the belief of player 2 in information set I is relevant - which is conditional to the partial history *h* = (R). Note that formally it is assumed that players have beliefs about their own strategies and payoffs, however, it seems appropriate to focus on the players' beliefs about their opponents.

A *rational* player i is assumed to be certain of his strategy and type and that he makes a best response in terms of beliefs about the other players. Moreover, in a dynamic model he might change his beliefs about the other players simply because he deviated from his own plan.

Battigalli and Siniscalchi (1999) constructed an epistemic model of the game in Figure 4. We will not reproduce this model here because this would necessitate to introduce a series of definitions and a quite voluminous formal apparatus. Instead we will give some intuitive interpretations of the model. First assume x > 1. If player 2 observes a partial history (R) then he should "believe" that player 1 is of payoff-type θ_{12} and choose u. This necessitates that (i) player 1 is of type θ_{12} , (ii) he "believes" that player 2 is rational and (iii) he "believes" that player 2 believes that he (player 1) is rational and "believes" in the rationality of player 2. Of course, this hierarchy of beliefs can be further expanded and formalized as *common certainty of opponent's rationality.* Only if player 2 assumes that his beliefs are consistent with the beliefs of player 1, including his belief that player 1 is rational, he can interpret the partial history (R) as a signal that player 1 is a θ_{12} payoff-type and will choose strategy u.

It is interesting to see that player 1 has no incentive to mislead player 2 in forming his beliefs. In case that player 1 is of type θ_{11}, and player 2 chooses u, both players are worse off. In this case, player 1 is even worse off than if he chooses strategy L.

Common certainty of an opponent's rationality refines our prediction about the behavior of player 1. This holds in every partial history of the game described by set H. Battigalli and Siniscalchi point out that the underlying argument has the flavor of *forward induction*. Let us assume player 2's initial belief (in partial history ϕ) was that player 1 is of type θ_{11}. Then he predicts that player 1 will select strategy L. If faced with partial history (R) and thus a deviation from his original prediction, player 2 attempts to find an explanation for this result that is consistent with the assumption that player 1 is rational. This implies that player 2 assigns a conditional probability of 1 to payoff-type θ_{12}, and results in his decision to pick strategy u. Given common certainty of opponent's rationality, player 1 anticipates that player 2 will choose u if he picks R so that R is optimal for partial history ϕ - i.e., at the beginning of the game.

If $x < 0$, however, then rationality and common certainty of an opponent's rationality lead to inconsistency. If player 1 is rational and is certain that player 2 is rational and certain about player 1's rationality in ϕ and (R), the player 1 should expect player 2 to choose d, and he should therefore choose L. Then, however, player 2 cannot be certain about player 2's rationality because the latter should pick L instead of R if he expects that player 2 chooses d.

4. 3 Rationalizable Strategies

Forward induction implies that a player seeks explanations of his opponent's observed behavior under the assumption that the latter is rational. Standard solution concepts can restrict the alternative explanations of a player and thus lead to outcomes that are, at least empirically, descriptively questionable. For instance, this applies to the subgame perfect equilibrium of the *centipede game* in Figure 5 which suggests that the first player chooses "down" (D) at his first more. However, if player 2 gets a chance to make a decision then this clearly indicates that player 1 does not follow the recipe of subgame perfectness. Now, player 2 could think that this is because player 1 does not understand the game or is not able to apply the *backward induction principle*, which subgame perfectness prescribes for finite games; or that player 1 wants to signal that he does not intend to follow the logic of subgame perfectness and thereby aim for a payoff larger than 1. Obviously, player 1 cannot hope for the payoff n if player 2 is payoff maximizing, since player 2 does not expect that he will have a chance to gain n+1while player 1 will receive 0 "with certainty" if the second player's last decision node is reached. Note that this is the round which backward induction considers first. Player 1 will reason player 2 will choose "down" in period n+1, and therefore choose "down" in round n if this period is ever reached. By the same reasoning, however, player 2 will choose "down" in round n-1, if he ever has a choice.

Now following the decision tree backwards from round n+1 to round 1, we see that the reasoning of rounds n+1 and n then applies to rounds 1 and 2. Consequently, backward induction and subgame perfectness prescribe that player 1 chooses "down" in the first period. Thus, if player 2 has the possibility to make a decision in round 2 he concludes that player 1 does not follow the backward induction principle or is a different type then assumed by the game tree - or player 1 assumes that player 2 is a different type. Then player 2 finds himself in a guessing game to which we apply the epistemic model.

Figure 5: The Centipede Game

The epistemic model amounts to looking for an epistemic type that *rationalizes* the opponents' actions. Because it is based on infinite hierarchies of conditional probabilities, it is not, therefore, very restrictive; in fact, it does not exclude any conceivable type of beliefs. It generalizes the idea of rationalizable strategies which was simultaneously introduced by Bernheim (1984) and Pearce (1984). In short, a strategy is rationalizable if it is a best reply to a strategy which is rationalizable. Obviously, the mutually best-reply strategies of a Nash equilibrium are rationalizable. However, rationalizable strategies are not necessarily equilibrium strategies as is straightforward from the example given in Figure 6 (which is borrowed from Bernheim (1984)).

Figure 6: Rationalizable strategies

	b_1	b_2	b_3	b_4
a_1	(0,7)	(2,5)	(7,0)	(0,1)
a_2	(5,2)	(3,3)	(5,2)	(0,1)
a_3	(7,0)	(2,5)	(0,7)	(0,1)
a_4	(0,0)	(0,-2)	(0,0)	(10,-1)

In the game of Figure 6, $\{a_1,a_2,a_3,a_4\}$ and $\{b_1,b_2,b_3,b_4\}$ are the sets of pure strategies of players 1 and 2, respectively. Note that strategy b_4 is never a best reply to any of the a-strategies of player 1. It is not, therefore, in the set of rationalizable strategies. However, we have to then see that a_4 is not rationalizable because it is not a best reply to any of the strategies in set $\{b_1,b_2,b_3\}$. The strategies in the sets $\{a_1,a_2,a_3\}$ and $\{b_1,b_2,b_3\}$ are rationalizable. The pair (a_2, b_2) is a Nash equilibrium and the corresponding strategies thus are mutually best replies. The remaining strategies in the sets $\{a_1, a_3\}$ and $\{b_1, b_3\}$ form a cycle of best replies: a_1 is a best reply to b_3 and b_3 is best reply to a_3 and a_3 is a best to b_1, while b_1 is a best reply to a_1, the latter relation closes the cycle.

The set of rationalizable strategies in a game with complete information is obtained by an iterative deletion procedure as applied above. In two-person games the outcome coincides with the result of iterated strict dominance. Rationalizability selects the strategies which are consistent with common certainty of rationality (Tan and da Costa Werlang, 1988). It is straightforward from the above example that beliefs can be consistent with strategies which are not mutually best replies, but are rationalizable.

4. 4 Epistemic Conditions for Nash Equilibrium

Aumann and Brandenburger (1995) discuss sufficient epistemic conditions for Nash equilibria. They show that, contrary a wide-spread belief in the game theoretical literature, common knowledge is not a necessary condition for Nash equilibrium. They prove the following theorem (Theorem A) for the 2-Player case: "Suppose that the game being played ..., the rationality of the players, and their conjectures are mutually known. Then the conjectures constitute a Nash equilibrium" (p.1162). - This is an interesting result, but the question is, do conjectures constitute a Nash equilibrium as stated in Theorem A? If we follow Aumann and Brandenburger and define a mixed strategy of a player as his or her *conjecture* which expresses the player's uncertainty about the other player's choices in the form of a probability distribution on the pure strategies of the other player, then it could be misleading to speak of a Nash equilibrium if the conjectures are *consistent* as the selected pure strategies are in general *not* mutually best replies to each other. The fact that player i's pure strategy concurs with i's conjecture of player j and is consistent j's conjecture, does not imply that the conditions for a Nash equilibrium are satisfied by the pure strategies chosen. There is a price if we assume that players do not randomize in a mixed strategy equilibrium but merely choose pure strategies which are consistent with their own conjectures about the choices of other players.

Aumann and Brandenburger point out that Theorem A does not call for common knowledge, which requires that all know, all know that all know, and so on ad infinitum. It is sufficient to know the other player's conjecture and his disposition to rationality to make a rational choice. Player i's choice of strategy x is rational if the conjecture of player j assigns a positive probability to x and x is optimal against i's conjecture (about j's strategy decision) and i's payoff function. Of course, i's conjecture depends on his knowledge of j's payoff function and j's disposition to rationality.

The formation of conjectures becomes more problematic if there are more then two players involved: players i and j can form different conjectures with respect to player k. Aumann and Brandenburger assume that players have *common priors* so that differences between conjectures are exclusively due to differences in their information. They prove Theorem B: "In an n-Player game, suppose that the players have common prior, that their payoff functions and their rationality are mutually known, and their conjectures are commonly known. Then for each player j, all other players i agree on the same conjecture σ_j about j; and the resulting profile (σ_1 ,...,σ_n) of mixed actions is a Nash equilibrium."

Interestingly, here the Nash equilibrium refers to mixed actions (where actions are pure strategies) which, of course, are assumed to concur with the conjectures. Does this mean that all players i assume that player j mixes his pure strategies in accordance with σ_j? The answer is not obvious from Aumann and Brandenburger, however, they point out that the Nash equilibrium in accordance with Theorem B merely needs common knowledge of the players' conjectures, not of the game or of the players' rationality. They also argue that the results in Theorem A and B are "tight" in the sense that none of the underlying conditions can be left out, or even significantly weakened. Nevertheless, as Polak (1999) shows, there is a possible trade-off between these conditions. Polak demonstrates that if we assume common knowledge of payoffs (i.e. complete information) then the common knowledge of rationality is implicit and we do not need to assume common priors to assure a Nash equilibrium in case of more than two players (see Theorem B).

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5. Experimental Research and Evolutionary Game Theory

We have seen that the formal apparatus of epistemic games is substantial, even for a rather simple game as in Figure 4. Modeling of infinite hierarchies of conditional probabilities is, in particular, "costly". Consequently, we can expect that the analysis of game situations will continue to make use of the Nash equilibrium and its refinements and other less sophisticated solution concepts. We have seen that these concepts do not always make sense and sometimes they describe outcomes that are far off what common sense predicts - and thus perhaps not very helpful to describe human behavior. They have, however, immensely helped us to increase our understanding of social interaction as, for instance, in the fields of industrial organization and of the emergence of norms.[15] Moreover, there is work which elaborates the epistemic conditions of solution concepts (see Aumann, 1995, and Aumann and Brandenburger, 1995) and thus may help to choose adequate solution concepts or to interpret the results of applying them adequately.

5. 1 Ultimatum Game and Fairness

Applying solution concepts concurs with our need to reduce complexity and to apply recipes in real-life decision situations. The implications of the interactive epistemology concept seem too demanding for real-world decision makers and even the application of less demanding solution concepts is often constrained by the human capacity for problem solving - the knowledge of appropriate skills and their applications and the ability to adequately interpret the results. In many cases, people neither analyze infinite hierarchies of beliefs nor choose that what the solution concepts prescribe. For instance, there has been a series tests of the so-called ultimatum game. The game is as follows: Players 1 and 2 are "invited" to share $100 subject that they agree how to divide this amount: if they do not

[15] See Aumann (1985) for this argument.

agree they get nothing. Player 1 is first to make a proposition how to share the money. Player 2's decision to accept or to reject this proposition concludes the game. (There is no second round or possibility of renegotiation, etc.).

The nicety of this game is that subgame perfectness describes a straightforward unique outcome: Player 1 will demand almost all of the money, and player 2 will accept a share which is as close to zero as the divisibility of the money allows. There is however ample empirical evidence that real-life decision-makers do not accept the crumbs assigned to them by this equilibrium if they are in the position of player 2. Experiments show that they ask for about one quarter of the money and reject the proposition of player if it offers less than this share.[16]

One explanation for this result is that the money does not represent the preferences such that the players' payoffs in the game are identical with money. Weibull (2000) maintains that experimental researchers have first of all to test for the players' payoff functions before they can claim that their experiments analyze the sharing of a fixed cake and test for subgame perfect outcomes. What if player 2's payoffs not only depend on his own share but also on the share of the other player or the distribution of the cake - and apply concepts of fairness to evaluate the outcome?

Now, if players apply notions of fairness to the ultimatum game then, of course, the subgame perfect equilibrium is an unlikely outcome even if payoffs are linear in money. Note that in this case we have two alternatives of modeling fairness: either as an argument of the payoff function, as implied by Weibull's proposal, or as the application of a solution concept which is different from subgame perfectness. It is well known that we can formulate strategy pairs such that any sharing proposition can be the outcome of a Nash equilibrium - which is, however, generally not subgame perfect (see, e.g., Rubinstein, 1982).

[16]See, e.g., Güth and Tietz (1990), Binmore et al. (1991), Güth et al. (1996), and Bolle (1990) for experimental results which falsify the subgame perfectness hypothesis.

Since Nash equilibrium strategies are rationalizable there should be an epistemic model implying a belief system which supports any specific sharing proposition. Alternatively, the application of a fairness norm can select a specific Nash equilibrium. This concurs with Binmore's (widely discussed) view that morality is merely a means of coordination: "Just as it is actually within our power to move a bishop like a knight when playing Chess, so we can steal, defraud, break promises, tell lies, jump lines, talk too much, or eat peas with our knives when playing a morality game. But rational folk choose not to cheat for much the same reason that they obey traffic signals" (1998, p.6). The problem here is that morality is not well defined, e.g., there are almost as many different notions of fairness as there are applications and it is not obvious how a society can refer to one of them to coordinate on a Nash equilibrium in a voluntary (and not exogenously enforceable) way.

5.2 Evolution and Descriptive Theory

Binmore suggests that we leave it to social evolution to develop consistent rules of fairness which can be applied to solve the coordination problems of a society. If it does not, and if there are competing societies which are more successful in solving the coordination problem, then the society is likely to "die out". Here, "dying out" could simply mean that the society chooses different rules and thus becomes a society different from the former one. However, history shows that, invaded by a competing society (i.e. a mutant), societies also literally vanish without transforming into another one.

On a personal level, evolution is synonymous with learning rather than genetic selection: applying new behavioral concepts, new ways of thinking and forms of social interaction. This can be result of deep insights or intensive observation of repeating phenomena, scientific studies, or simply imitating other individuals who are generally more successful - instead of calculating best replies. In this case, "players need know only what was successful, not why it was successful" (Mailath, 1998, p. 1355). Successful behavior thus becomes more prevalent not only because, for instance, market competition selects against "unprofitable" behavior, but also because agents imitate successful behavior. However, imitating successful behavior is not always a successful pattern of behavior: a market might be large enough for two suppliers but overcrowded if there are three.[17]

In a recent paper, Selten and Ostmann (2001) introduce the concept of an imitation equilibrium. The underlying model consists of a normal form game G and a *reference structure* R. The latter assigns a set R(i) of players to each player i of game G which contains those players "who are sufficiently similar to be imitated if they are more successful" (p. 113). A second constituent element of the model is the strategy of *explorative deviations* of those players who have profits after imitation which are at least as high as the highest in their reference groups. Then, an *imitation equilibrium* is characterized by a strategy vector which (a) contains no opportunities of profitable imitations and (b) and is stable against exploratory deviations. Of course, an imitation equilibrium depends (a) on the *reference structure* - whether it is universal such that the reference set of a player is always the set of all other players, or constrained to subsets - and (b) on the available strategies of exploratory deviations: a global imitation equilibrium requires stability against *any* exploratory deviation.

Contrary to genetic evolution, the selection mechanism ("market") in social evolution is, at least to some extent, subject to the players' discretion and therefore possibly under the influence of successful social groups or individuals who are then able "to define the rules of the game". Moreover, social evolution does not necessarily follow the dynamic patterns of genetic evolution studied in

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[17]Sherman and Willett (1967) contains a market model, which can be modeled as volunteer's dilemma (see Diekmann, 1985), in which the threat of potential entry and overcrowding, resulting in losses, discourages entry if suppliers choose their maximin strategy.

biology: learning can be fast, especially when it is in the form of imitation, and communication can allow for almost immediate changes on a large scale.

In situations where we lack sufficient observations, learning has to refer to descriptive theories of game behavior if the decision situation is characterized by strategic interaction. Experimental studies deliver building blocks for such a theory: "Typically, such descriptive theories formalize ideas on limited rationality decision-making. Satisficing and the adaptation of aspiration levels rather than optimization play an important role" (Selten, 1987, pp. 84f.).

5. 3 An Evolutionary Game

Accordingly, evolutionary game theory takes an equilibrium as the outcome of adaptation (or learning) and selection rather than as the result of strategic reasoning by rational agent. This idea has been formalized by two sets of solution concepts: static equilibrium concepts, the evolutionary stable strategies (ESS) and its variants, which are closely related to the Nash equilibrium, and a set of dynamic concepts which examine the stability of the evolutionary process (asymptotic stability or Liapunov stability). In the latter case, the dynamics of the process is often described by the standard replicator equation which says that the growth of state variable x_i^t (in time t) is defined by the difference between its fitness, $f(i,x^t)$, and the average fitness of the population, $f(x^t,x^t)$ where x^t is a vector which has the elements x_i^t with $i = 1,...,n$. Here, x_i^t can be the share of a social group in a population, or the share of a strategy (or a mode of behavior) picked by a set of players. In the latter case, fitness can be identified by payoffs.

The formal expression of the replicator function is:

(R) $$dx_i^t/dt = x_i^t\,[f(i,x^t) - f(x^t,x^t)]$$

A state is a *rest point* if the state variables x_i^t, i = 1, ...,n, do not change, i.e., if $dx_i^t/dt = 0$ holds for all derivatives with respect to time t. A rest point is *Liapunov stable* if the dynamic process does not take states which are close to the rest point far away in case that the system is destabilized by a external shock. A rest point is *asymptotically stable* if any dynamically process which starts sufficiently close to the rest point converges to this rest point. The specification of closeness decides whether the dynamic model is globally or only locally asymptotically stable. If the basin of attraction covers the full domain of each state variable, then the dynamic system is globally stable.

Figure 6: Rest Points and Liapunov Stability

		Player 2	
		L	R
Player 1	T	(1,1)	(1,0)
	B	(1,1)	(0,0)

We can illustrate some of these concepts by means of the game in Figure 6.[18] First, we can observe that the game has two obvious Nash equilibria in pure strategies, however, any mixed strategy on T and B is part of a Nash equilibrium if player 2 picks L. But all Nash equilibria of this game are weak. To prepare for the evolutionary interpretation of the game, we assume that p is the share of strategy T in the population of player 1: either we assume that population 1 consists of p players of type 1 who pick T and (1-p) players of this type who pick B, or we think of type 1 players who randomize on picking T and B with probabilities p and (1-p). Correspondingly, q represents the probability (or share) of L in population 2.

We now describe the possible development of p^t and q^t by means of replicator functions of the type in (R). We get:

[18]The game is in Samuelson and Zhang (1992); we follow its use and interpretation in Mailath (1998).

(R1) $dp^t/dt = p^t (1-p^t)(1- q^t)$ and

(R2) $dq^t/dt = q^t (1- q^t)$

These two differential equations indicate that the system has a rest point for $q^t = 1$; irrespective of p^t, i.e., the composition or behavior of population 1. If $q^t = 1$ both (R1) and (R2) are zero, i.e., nothing will change as long as there is no external shock such that an agent chooses a different strategy. Moreover, the system will not be destabilized if there is an external shock so that p^t changes as long $q^t = 1$. However, what happens if "suddenly" some members of population 2 choose R instead of L such that $q^t < 1$. (Of course, this change in behavior is not "rational" because the R strategy is strictly dominated by L.) As a consequence we expect from (R2) that the q^t will grow till $q^t = 1$is achieved and, again, all members of population 2 pick L. Moreover, we expect from (R1) that p^t , the share of members of population 1 who pick T, will increase till either $q^t = 1$ or $p^t = 1$ or both conditions are satisfied. If p^t is far away from $p^t = 1$ while q^t is close to $q^t = 1$, since the shock which lead to $q^t <1$ was small, then we expect that $q^t = 1$ is reached before $p^t = 1$and there will be a new rest point $(p^{t'},1)$ close the combination $(p^t,1)$ which characterized the earlier rest point (with t' > t). Thus the rest points of the dynamic system described by (R1) and (R2) are Liapunov stable. However, they are not asymptotically stable because the derivatives in (R1) and (R2) are nonnegative and external shocks will always lead to an increase of p^t until $p^t =1$ is reached. As a consequence, an earlier rest point can never be reached after a perturbation which affected q^t. The exception seems to be with state $p^t =1$ and $q^t = 1$. A shock in q^t , which implies that parts of the population 2 play R, induces a growth of q^t till $q^t = 1$ is reached while p^t (= 1) will not change. However, if there is (only) a perturbation in p^t such $p^t < 1$ and $q^t = 1$, then it follows from the replicator functions that no forces exist that bring p^t back to $p^t = 1$. Thus, the state $p^t = 1$ and $q^t = 1$ is not asymptotically stable.

5.4 Evolutionary Solution Concepts

For the analysis of an evolutionary game the dynamics is often of interest in order to see whether the game describes a stable environment: Liapunov stable or even asymptotically stable. If not, the social or political controller (e.g., government) might consider to modify the game, possibly by changing the payoffs. Evolutionary game also offers static concepts, like ESS, to analyze stability problems, which are inherently dynamic, without explicitly analyzing dynamics. A (monomorphic) population is evolutionary stable if it satisfies the following condition (ESS) if a "mutant strategy" m invades the established population described by strategy s.

(ESS) There is a $\varepsilon°$ such that for every $\varepsilon < \varepsilon°$:

$$(1-\varepsilon)f(s,s) + \varepsilon f(s,m) > (1-\varepsilon)f(m,s) + \varepsilon f(m,m)$$

Note that the left-hand side of this inequality represents the expected fitness of the "established strategy" s, where f(s,s) is the fitness of s when meeting a strategy of the same kind and f(s,m) is the fitness of s when meeting a mutant strategy. The right-hand side represents the expected fitness of the mutant m where f(m,s) and f(m,m) express the fitness of m when m meets s and m, respectively.

The value of $\varepsilon°$ can be very small, in any case, the inequality condition has to hold for all $\varepsilon < \varepsilon°$. Thus, if we find a very small $\varepsilon°$, so that entry of m is only "marginal", and the inequality condition in ESS holds, then the population s is evolutionary stable. ESS implies that

(i) $f(s,s) \geq f(m,s)$ and

(ii) if f(s,s) = f(m,s) then f(s,m) > f(m,m)

It is straightforward from (i) that ESS implies a Nash equilibrium where m represents the possible deviation. It is also straightforward that there are Nash equilibria which do not satisfy ESS, e.g., if f(s,s) = f(s,m) = f(m,m) holds for all and the Nash equilibrium (s,s) is weak.

Static solution concepts to evolution seem especially appropriate when the dynamics of social evolution is not continuous and the replicator functions are inappropriate. I have argued above that learning can be fast and communication can allow for almost immediate changes on a large scale. It therefore seems appropriate to consider values of $\varepsilon°$ which are non-marginal when we apply ESS to human interaction. For example, Peters (1997) discusses a modified ESS-concept for the analysis of larger invasions by making $\varepsilon°$ an exogenous parameter. He applies this concept to the emergence of standards where critical mass effects and threshold values are substantial and communication and learning are essential.

There is a strong relationship between static solution concepts, such as ESS and Nash equilibrium, the stability results of the dynamic analysis captured by the replicator function.[19] For example, if a state x^t is asymptotically stable then it coincides with a Nash equilibrium. If x^t satisfies ESS then it is asymptotically stable if the continuous replicator dynamics apply. However, asymptotically stable rest points of the replicator dynamics do not necessarily satisfy ESS, e.g, if some of the mutants are considered with zero probabilities, only.

If a game is asymmetric, like in Figure 6, then a Nash equilibrium is asymptotically stable if and only if it is strict (which is not the case in the game of Figure 6) and the dynamics is described by replicator functions. However, if a game is asymmetric then ESS does not exist. ESS exists for the symmetrized

[19] Results are summerized, e.g. in Mailath (1998).

game, where players have equal chances to be in the alternative roles which imply the asymmetry, if a strict Nash equilibrium exists for this game. From this we can conclude that ESS is a more rigorous concept for testing stability than asymptotic stability while the latter is more rigorous in this test than the Nash equilibrium concept. The Nash equilibrium is, however, more rigorous than the rest point concept. For instance, if only strategy R will be played in the game of Figure 6, so that $q^t = 0$ then the state $p^t = 1$ and $q^t = 0$ is a rest point, but not an equilibrium.

These results can be directly applied to problem of equilibrium selection which, however, is only relevant if one works in this category. We have outlined a series of arguments above which do not necessarily support this approach. But the evolutionary approach does give a nice interpretation of mixed strategy by means of introducing heteromorphous populations with members who characterized by different pure strategies. (See the game in Figure 6 above.) Unfortunately, so far no systematic analysis exists which relates the formation of beliefs with the evolutionary approach. Are we more successful, i.e., more likely to survive, if we solve the epistemic game for infinite levels of belief formation? Or should we simply be more optimistic about the beliefs which other players have about ourselves?

6. Final Remarks

The hypothesis of the above brief history of game theory is that the various stages of its development are the result of different assumptions on the nature of the decision makers, i.e., of the "image of man" underlying the various approaches. This explains why I did not discuss cooperative game theory which is characterized by the assumption that the players can make binding (and enforceable) agreements. As a consequence, the coordination of strategies and solving conflicts are no problem if player want to solve them. What remains is the question on what result the players will agree.[20]

Of course, this brief history has omitted many issues of game theory but my intention was not to give a complete overview - only to discuss the changes in "style" and to relate them to changes in the "image of man".

[20]See, e.g., Owen (1995) for the Nash solution (pp. 190-197), the core and stable sets (pp. 218-232) and related solution concepts of cooperative game theory.

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